Online Appendix: Random Inspections and Periodic Reviews: Optimal Dynamic Monitoring

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A Analysis of Relaxed Problem using Dynamic Programming

As an intermediate step toward characterizing the optimal policy in the general model, we study a relaxed problem that ignores the agent's incentive constraint. When $u(\cdot)$ is convex, and both the cost of monitoring c and effort k are small enough, the solution of such a relaxed problem satisfies the agent's incentive constraint being thus the optimal policy.[1] Moreover, even if moral hazard is severe, the trade-offs identified in the unconstrained problem influence the structure of the optimal policy.

Without incentive constraints, it is convenient to analyze the problem using dynamic programming. Consider the evolution of reputation between two inspection dates. Given that the firm exerts full effort, $a = \bar{a}$, the reputation between two inspections dates evolves according to

$$\dot{x}_t = \lambda(\bar{a} - x_t) \tag{A.1}$$

The optimal policy is Markovian in reputation. Denoting by $\mathcal{A}$ the set of reputations that lead to immediate inspection, the principal payoff given beliefs x, which we denote by $U(x)$, solves the Hamilton-Jacobi-Bellman (HJB) equation

$$rU(x) = u(x) + \lambda(\bar{a} - x)U'(x),\ x \notin \mathcal{A} \tag{A.2a}$$

$$U(x) = xU(1) + (1 - x)U(0) - c,\ x \in \mathcal{A}. \tag{A.2b}$$

We can guess and then verify that the optimal policy is given by an audit set $\mathcal{A} = [\underline{x}, \overline{x}]$, where

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[1] The solution of this relaxed problem also characterizes the optimal policy when effort (but not quality) is observable (recall we assume $u'(0) \geq 1$ so full effort is optimal in the first best).

$\underline{x} \le \bar{a} \le \overline{x}$, where it can be shown that the threshold $\hat{x} \in \{\underline{x}, \overline{x}\}$ satisfies the boundary conditions:

$$U(\hat{x}) = \hat{x}U(1) + (1-\hat{x})U(0) - c \tag{A.3a}$$

$$U'(\hat{x}) = U(1) - U(0). \tag{A.3b}$$

Hence, we have the following standard result:

Result A.1 (Benchmark)**.** *Suppose that U is a function satisfying the HJB equation* (A.2a)-(A.2b) *together with the boundary conditions* (A.3a)-(A.3b)*. Then U is the value function of the Principal's optimization problem and the optimal policy is to monitor the firm whenever $x_t \in \mathcal{A} = [\underline{x}, \overline{x}]$.*

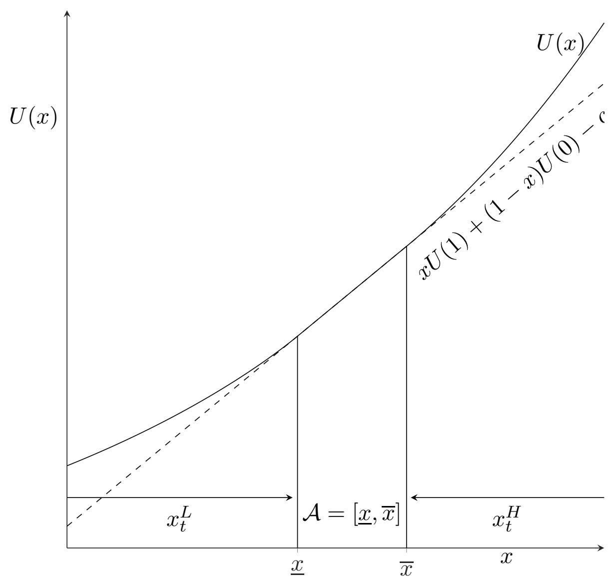


Figure 1: Value Function. The optimal policy requires to monitor whenever reputation enters the audit set, $x_t \in \mathcal{A}$.

Figure 1 illustrates the principal's payoff as a function of beliefs. Observe that after an inspection beliefs reset to either $x = 0$ or $x = 1$ because reviews are fully informative. Then, beliefs begin to drift deterministically toward $\bar{a}$, which lies in the interior of the audit set $\mathcal{A}$. When beliefs hit the boundary of $\mathcal{A}$, the principal monitors the firm for certain. Naturally, the principal acquires information when enough uncertainty has accumulated, namely when the distance between $U(x)$ and the line connecting $U(0)$ and $U(1)$ gets large and when beliefs get close to $\bar{a}$, so the drift in beliefs is small.

The size of the monitoring region $\mathcal{A}$ depends on the convexity of the principal's objective function and the cost of monitoring c since these parameters capture the value and cost of information, respectively. In the extreme case when $u(\cdot)$ is linear (or c is too large) the optimal policy is never to monitor the firm but let beliefs converge to $\bar{a}$ (but of course in this case the incentive constraint would be violated since there are no rewards to effort in the absence of monitoring). By contrast, as $u(\cdot)$ becomes more convex, the monitoring region widens, leading to more frequent monitoring.

Eventually, the incentive constraint becomes slack, which, as mentioned above, implies that the solution to the relaxed problem is the optimal monitoring policy.

Figure 1 illustrates the optimal policy as a function of beliefs. Notice that between inspection dates beliefs evolve deterministically and monotonically over time, hence there is an equivalent representation of the monitoring policy based upon the time since last review, $t - T_n$, and the outcome observed in the last review, θ_{T_n}. Specifically, define:

$$\tau_H \equiv \inf\{t : x_t = \overline{x}, x_0 = 1\} = \frac{1}{\lambda}\log\left(\frac{1-\bar{a}}{\overline{x}-\bar{a}}\right)$$

$$\tau_L \equiv \inf\{t : x_t = \underline{x}, x_0 = 0\} = \frac{1}{\lambda}\log\left(\frac{\bar{a}}{\bar{a}-\underline{x}}\right).$$

We can then represent the policy by the $n_{th}-$monitoring time as $T_n = T_{n-1} + \tau_{\theta_{T_{n-1}}}$.[2]

Remark A.2. *This representation of the optimal monitoring policy applies to the case in which both τ_L and τ_H are finite. Depending on the specific parameters of the model, either τ_L or τ_H can be infinite, or in other words, there is no further monitoring after some outcomes. In terms of the policy specified as a function of beliefs this means that either $\underline{x} = \bar{a}$ or $\overline{x} = \bar{a}$. In this case, the value matching and smooth pasting conditions are only valid at the threshold that is different from $\bar{a}$.*

A.1 Proof of Result A.1

Proof. Differentiating the HJB equation we get that for any $x \notin [\underline{x}, \overline{x}]$ we have

$$(r+\lambda)U'(x) = u'(x) + \lambda(\bar{a} - x)U''(x) \tag{A.4a}$$

$$(r+2\lambda)U''(x) = u''(x) + \lambda(\bar{a} - x)U'''(x) \tag{A.4b}$$

Using (A.4b) we get that for any $x > \bar{a}$ we have $U''(x) = 0 \Rightarrow U'''(x) > 0$. This means that $U''(\overline{x}) \geq 0 \Rightarrow U''(x) > 0$ for all $x > \overline{x}$. Similarly, for any $x < \bar{a}$ we have $U''(x) = 0 \Rightarrow U'''(x) < 0$ which means that $U''(\underline{x}) \geq 0 \Rightarrow U''(x) > 0$ for all $x < \underline{x}$. Evaluating (A.4a) at $\overline{x}$ and using the smooth pasting condition we find that

$$(r+\lambda)(U(1) - U(0)) = u'(\overline{x}) + \lambda(\bar{a} - \overline{x})U''(\overline{x})$$

Hence, U we have that $U''(\overline{x}) \geq 0$ and $U''(\underline{x}) \geq 0$ if and only if

$$\frac{u'(\underline{x})}{r+\lambda} \leq U(1) - U(0) \leq \frac{u'(\overline{x})}{r+\lambda} \tag{A.5}$$

[2]The only exception would be the case when $x_0 \in (0,1)$. In this case $T_1 = \frac{1}{\lambda}\log\left(\frac{x_0-\bar{a}}{\overline{x}-\bar{a}}\right)$ if $x_0 > \overline{x}$; $T_1 = \frac{1}{\lambda}\log\left(\frac{x_0-\bar{a}}{\underline{x}-\bar{a}}\right)$ if $x_0 < \underline{x}$ and $T_1 = 0$ otherwise. After T_1, the policy would be the one described in the text.

The HJB equation together with the boundary conditions imply that

$$r(U(0) + \overline{x}(U(1) - U(0))) = u(\overline{x}) + \lambda(\bar{a} - \overline{x})(U(1) - U(0))$$
$$r(U(0) + \underline{x}(U(1) - U(0))) = u(\underline{x}) + \lambda(\bar{a} - \underline{x})(U(1) - U(0))$$

Taking the difference between these two equations and rearranging terms we find that

$$U(1) - U(0) = \frac{1}{r+\lambda} \frac{u(\overline{x}) - u(\underline{x})}{\overline{x} - \underline{x}}.$$

It follows from the convexity of u that inequality (A.5) is satisfied. The fact that U is increasing follows directly from the convexity of U and equation (A.4a).

Next, let's define

$$H(x) \equiv xU(1) + (1-x)U(0) - U(x).$$

The convexity of U implies that H is concave and $H(x) = c$ for $x \in [\underline{x}, \overline{x}]$ and $H(x) < c$ for $x \notin [\underline{x}, \overline{x}]$. Hence, we get that

$$xU(1) + (1-x)U(x) - U(x) \leq c. \tag{A.6}$$

Similarly, let's define

$$G(x) \equiv u(x) + \lambda(\bar{a} - x)(U(1) - U(0)) - r(xU(1) + (1-x)U(0) - c).$$

Differentiating the previous equation twice we get that $G''(x) = u''(x) > 0$. Because $U(\cdot)$ is continuously differentiable we have that $G(\underline{x}) = G(\overline{x}) = 0$. Hence, we can conclude that $G(x) < 0$ for all $x \in (\underline{x}, \overline{x})$. Accordingly,

$$0 \geq u(x) + \lambda(\bar{a} - x)U'(x) - rU(x), \;\; x \in [0,1]. \tag{A.7}$$

The final step is to verify that we can not improve the payoff using an alternative policy. Let $(\tilde{T}_n)_{n\geq 1}$ and let $\tilde{x}_t$ be the belief process induce by this policy. Applying Ito's lemma to the process $e^{-rt}U(\tilde{x}_t)$ we get

$$\begin{aligned} e^{-rt}E[U(\tilde{x}_t)] &= U(x_0) + E\left[\int_0^t e^{-rs}(\lambda(\bar{a} - \tilde{x}_t)U'(\tilde{x}_t) - rU(\tilde{x}_t))\mathrm{d}s + \sum_{s\leq t} e^{-rs}\big(\tilde{x}_s U(1) + (1-\tilde{x}_s)U(0) - U(\tilde{x}_s))\right] \\ &\leq U(x_0) - E\left[\int_0^t e^{-rs}u(\tilde{x}_t)\mathrm{d}s - \sum_{s\leq t} e^{-rs}c\right], \end{aligned} \tag{A.8}$$

where we have used inequalities (A.6) and (A.7). Taking the limit when $t \to \infty$ we conclude that

$$U(x_0) \geq E\left[\int_0^{\infty} e^{-rs}u(\tilde{x}_t)\mathrm{d}s - \sum_{\tilde{T}_n \geq 0} e^{-r\tilde{T}_n}c\right]$$

The proof concludes noting that (A.8) holds with equality for the optimal policy. □

B Linear Case: Alternative Proof of Proposition 3

Proof. Let T be the first monitoring time so the principal's cost at time zero satisfies the recursion

$$C_0 = E_0[e^{-rT}](c + C_0)$$

and the incentive compatibility constraint at time zero is

$$E_0[e^{-(r+\lambda)T}] \geq \underline{q}$$

We show that if there is any time τ such that the incentive compatibility constraint is slack, then we can find a new policy that satisfies the IC constraint and yields a lower expected monitoring cost to the principal. In fact, it is enough to show that if the IC constraint is slack at some time $\tilde{\tau}$ then we can find an alternative policy that leaves $E_0[e^{-(r+\lambda)T}]$ unchanged at time zero, remains IC at $\tau > 0$ and reduces $E_0[e^{-rT}]$. We only consider the case in which there is positive density just before $\tilde{\tau}$ as the argument for the case in which there is an atom at $\tilde{\tau}$ and zero probability just before $\tilde{\tau}$ is analogous. Suppose the IC constraint is slack at time $\tilde{\tau}$ and let $\tau^{\dagger} = \sup\{\tau < \tilde{\tau} : \text{IC constraint binds}\}$: such a date must exist as otherwise we could postpone somewhat all inspection times before $\tilde{\tau}$ and still satisfy all IC constraints (obviously saving costs). Moreover, we can assume without loss of generality that $\tau^{\dagger} = 0$. Suppose the monitoring distribution $F(\tau)$ is such that $f(\tau) > 0$ for some interval $(\tilde{\tau} - \epsilon, \tilde{\tau})$, then we can find small ϵ_0 and η and construct an alternative monitoring distribution $\hat{F}(\tau)$ that coincides with $F(\tau)$ outside the intervals $(0, \epsilon_0)$ and $(\tilde{\tau} - \epsilon_0, \tilde{\tau} + \epsilon_0)$. For any $\tau \in (\tilde{\tau} - \epsilon_0, \tilde{\tau})$ the density of the alternative policy is

$$\hat{f}(\tau) = f(\tau) - \eta,$$

while for $\tau \in (0, \epsilon_0)$ it is

$$\hat{f}(\tau) = f(\tau) + \alpha\eta,$$

and for $\tau \in (\tilde{\tau}, \tilde{\tau} + \epsilon_0)$ it is

$$\hat{f}(\tau) = f(\tau) + (1 - \alpha)\eta.$$

We can pick $\alpha \in (0,1)$ such that IC constraint is not affected at $\tau = 0$, that is $\alpha \in (0,1)$ satisfies

$$\alpha \int_0^{\epsilon_0} e^{-(r+\lambda)\tau} \mathrm{d}\tau + (1-\alpha) \int_{\tilde{\tau}}^{\tilde{\tau}+\epsilon_0} e^{-(r+\lambda)\tau} \mathrm{d}\tau - \int_{\tilde{\tau}-\epsilon_0}^{\tilde{\tau}} e^{-(r+\lambda)\tau} \mathrm{d}\tau = 0,$$

and we can pick ϵ_0 and η small enough so that the IC constraint still holds for all $\tau > 0$. Because the IC constraint is not affected at $\tau = 0$ we have that

$$\int_0^{\infty} e^{-(r+\lambda)\tau} \mathrm{d}F(\tau) = \int_0^{\infty} e^{-(r+\lambda)\tau} \mathrm{d}\hat{F}(\tau).$$

Define the random variable $z \equiv e^{-(r+\lambda)\tau}$, and let G and $\hat{G}$ be the respective CDFs of z. We have that

$$\int_0^1 z \mathrm{d}G(z) = \int_0^1 z \mathrm{d}\hat{G}(z).$$

By construction $G(z)$ and $\hat{G}(z)$ have same mean and cross only once which means that $\hat{G}(z)$ is a mean-preserving spread of $G(z)$. Noting that

$$\int_0^{\infty} e^{-r\tau} \mathrm{d}F(\tau) = \int_0^1 z^{\frac{r}{r+\lambda}} \mathrm{d}G(z),$$

where $z^{r/(r+\lambda)}$ is a strictly concave function, and using the fact that $\hat{G}(z)$ is a mean-preserving spread of $G(z)$, we immediately conclude that

$$\int_0^1 z^{\frac{r}{r+\lambda}} \mathrm{d}\hat{G}(z) < \int_0^1 z^{\frac{r}{r+\lambda}} \mathrm{d}G(z),$$

and so the monitoring distribution $\hat{F}(\tau)$ yields a lower cost of monitoring: This contradicts the optimality of $F(\tau)$ and implies that the optimal policy must be such the IC constraint binds at all time, hence it is given by a constant monitoring rate m^*. □

C Comparative Statics

C.1 Proof of Proposition 4

Comparative static c: Let G_{det} and G_{rand} be the maximization problems in the operators above so we write the optimization in the fixed point problem as

$$\max_{\alpha\in[0,1]} \alpha G_{\text{rand}} + (1-\alpha)G_{\text{det}}$$

We can fix the continuation values and show that we have single crossing in (c, U_H, U_L). In the previous expressions, we have that

$$\begin{aligned}
\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial(-c)} &= e^{-r\hat{\tau}}\left(\frac{r}{r+\lambda\underline{q}}e^{(r+\lambda)\hat{\tau}}\underline{q} + \frac{\lambda\underline{q}}{r+\lambda\underline{q}}\right) - e^{-r\bar{\tau}} \\
&= e^{\lambda\hat{\tau}}\underline{q}\left(\frac{r}{r+\lambda\underline{q}} + e^{-(r+\lambda)\hat{\tau}}\frac{\lambda}{r+\lambda\underline{q}}\right) - e^{-(r+\lambda)\bar{\tau}}e^{\lambda\bar{\tau}} \\
&\leq e^{\lambda\hat{\tau}}\underline{q} - e^{-(r+\lambda)\tau_{\text{bind}}}e^{\lambda\bar{\tau}} \\
&= \left(e^{\lambda\hat{\tau}} - e^{\lambda\bar{\tau}}\right)\underline{q}
\end{aligned}$$

which is negative if $\hat{\tau} < \bar{\tau}$. Next, we have that

$$\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial U_H} = e^{-r\hat{\tau}}\left[\left(\frac{e^{(r+\lambda)\hat{\tau}}-1}{1-\underline{q}}\right)\underline{q}x^{\theta}_{\hat{\tau}} + \left(\frac{1-e^{(r+\lambda)\hat{\tau}}\underline{q}}{1-\underline{q}}\right)\int_{\hat{\tau}}^{\infty} e^{-(r+m)(\tau-\hat{\tau})}mx^{\theta}_{\tau}\mathrm{d}\tau\right] - e^{-r\bar{\tau}}x^{\theta}_{\bar{\tau}}$$

If we replace

$$\begin{aligned}
\int_{\hat{\tau}}^{\infty} e^{-(r+m)(\tau-\hat{\tau})}x^{\theta}_{\tau}\mathrm{d}\tau &= \frac{\bar{a}}{r+m} + \frac{x^{\theta}_{\hat{\tau}} - \bar{a}}{r+\lambda+m} \\
\int_{\hat{\tau}}^{\infty} e^{-(r+m)(\tau-\hat{\tau})}(1-x^{\theta}_{\tau})\mathrm{d}\tau &= \frac{1-\bar{a}}{r+m} - \frac{x^{\theta}_{\hat{\tau}} - \bar{a}}{r+\lambda+m},
\end{aligned}$$

and after some tedious simplifications we obtain

$$\begin{aligned}
\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial U_H} &= e^{-r\hat{\tau}}\left[(1-\underline{q})e^{(r+\lambda)\hat{\tau}}\frac{m}{r+\lambda}x^{\theta}_{\hat{\tau}} + \left(1-e^{(r+\lambda)\hat{\tau}}\underline{q}\right)\frac{\lambda(1-\underline{q})}{(r+\lambda\underline{q})(r+\lambda)}m\bar{a}\right] - e^{-r\bar{\tau}}x^{\theta}_{\bar{\tau}} \\
&= e^{\lambda\hat{\tau}}\underline{q}x^{\theta}_{\hat{\tau}} + \left(e^{-r\hat{\tau}} - e^{\lambda\hat{\tau}}\underline{q}\right)\frac{\lambda\underline{q}}{r+\lambda\underline{q}}\bar{a} - e^{-r\bar{\tau}}x^{\theta}_{\bar{\tau}}
\end{aligned}$$

Noticing that

$$e^{\lambda\tau}x^{\theta}_{\tau} = \theta + \bar{a}(e^{\lambda\tau}-1),$$

we obtain

$$
\begin{aligned}
\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial U_H} &= e^{\lambda\hat{\tau}}\underline{q}x^{\theta}_{\hat{\tau}} + \left(e^{-r\hat{\tau}} - e^{\lambda\hat{\tau}}\underline{q}\right)\frac{\lambda\underline{q}}{r+\lambda\underline{q}}\bar{a} - e^{-r\bar{\tau}}x^{\theta}_{\bar{\tau}} \\
&= \underline{q}\left(\theta - \bar{a}\right) + \left[e^{\lambda\hat{\tau}}\frac{r}{r+\lambda\underline{q}} + e^{-r\hat{\tau}}\frac{\lambda}{r+\lambda\underline{q}}\right]\underline{q}\bar{a} - e^{-(r+\lambda)\bar{\tau}}\left(\theta + \bar{a}(e^{\lambda\bar{\tau}}-1)\right) \\
&\leq \underline{q}\left(\theta - \bar{a}\right) + \left[e^{\lambda\hat{\tau}}\frac{r}{r+\lambda\underline{q}} + e^{-r\hat{\tau}}\frac{\lambda}{r+\lambda\underline{q}}\right]\underline{q}\bar{a} - \underline{q}(\theta-\bar{a}) - e^{-r\bar{\tau}}\bar{a} \\
&= \left[e^{\lambda\hat{\tau}}\frac{r}{r+\lambda\underline{q}} + e^{-r\hat{\tau}}\frac{\lambda}{r+\lambda\underline{q}}\right]\underline{q}\bar{a} - e^{-r\bar{\tau}}\bar{a}
\end{aligned}
$$

The last expression is increasing in $\hat{\tau}$, which means that if $\hat{\tau} \leq \bar{\tau}$ then

$$
\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial U_H} \leq -e^{\lambda\bar{\tau}}\left(e^{-(r+\lambda)\bar{\tau}} - \underline{q}\right)\frac{r\bar{a}}{r+\lambda\underline{q}} \leq 0,
$$

where the last inequality follows from the IC constraint. We can repeat the same calculations for U_L.

$$
\begin{aligned}
\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial U_L} &= e^{-r\hat{\tau}}\left[\left(\frac{e^{(r+\lambda)\hat{\tau}}-1}{1-\underline{q}}\right)\underline{q}(1-x^{\theta}_{\hat{\tau}}) + \left(\frac{1-e^{(r+\lambda)\hat{\tau}}\underline{q}}{1-\underline{q}}\right)\int_{\hat{\tau}}^{\infty} e^{-(r+m)(\tau-\hat{\tau})}m(1-x^{\theta}_{\tau})\mathrm{d}\tau\right] \\
&\qquad - e^{-r\bar{\tau}}(1-x^{\theta}_{\bar{\tau}}) \\
&= e^{\lambda\hat{\tau}}\underline{q}(1-x^{\theta}_{\hat{\tau}}) + \left(e^{-r\hat{\tau}} - e^{\lambda\hat{\tau}}\underline{q}\right)\frac{\lambda\underline{q}}{r+\lambda\underline{q}}(1-\bar{a}) - e^{-r\bar{\tau}}(1-x^{\theta}_{\bar{\tau}})
\end{aligned}
$$

Replacing

$$
1 - x^{\theta}_{\tau} = e^{-\lambda\tau}(1-\theta) + (1-e^{-\lambda\tau})(1-\bar{a})
$$

we get that

$$
\begin{aligned}
\frac{\partial(G_{\text{rand}} - G_{\text{det}})}{\partial U_L} &= \underline{q}(\bar{a}-\theta) + \left(e^{\lambda\hat{\tau}}\underline{q}\frac{r}{r+\lambda\underline{q}} + e^{-r\hat{\tau}}\frac{\lambda\underline{q}}{r+\lambda\underline{q}}\right)(1-\bar{a}) - e^{-r\bar{\tau}}(1-x^{\theta}_{\bar{\tau}}) \\
&= \underline{q}(\bar{a}-\theta) + \left(e^{\lambda\hat{\tau}}\underline{q}\frac{r}{r+\lambda\underline{q}} + e^{-r\hat{\tau}}\frac{\lambda\underline{q}}{r+\lambda\underline{q}}\right)(1-\bar{a}) - e^{-(r+\lambda)\bar{\tau}}\left(\bar{a} - \theta + e^{\lambda\bar{\tau}}(1-\bar{a})\right) \\
&\leq \left[e^{\lambda\hat{\tau}}\frac{r}{r+\lambda\underline{q}} + e^{-r\hat{\tau}}\frac{\lambda}{r+\lambda\underline{q}}\right]\underline{q}(1-\bar{a}) - e^{-r\bar{\tau}}(1-\bar{a}) \\
&\leq 0
\end{aligned}
$$

where the last inequality follows if $\hat{\tau} \leq \bar{\tau}$ by the same reason as in the case of U_H. Hence, in order to verify single crossing in $(-c, U_L, U_H)$ it is enough to show that $\hat{\tau} \leq \bar{\tau}$. Notice that, for a given continuation value (U_L, U_H), the solution to the deterministic problem, $\bar{\tau}$, is increasing in c, and that whenever $\bar{\tau} < \tau^{\text{bind}}$ (so the IC constraint is slack), the solution to the optimal control problem must be $\bar{\tau}$. Let $c^{\dagger} = \sup\{c \geq 0 : \bar{\tau} < \tau^{\text{bind}}\}$, so for any $c < c^{\dagger}$ the solution for a given continuation value (U_L, U_H) is $\bar{\tau}$. On the other hand, for any $c \geq c^\dagger$ we have that $\bar{\tau} = \tau^{\text{bind}} \geq \hat{\tau}$, which means that $G_{\text{rand}} - G_{\text{det}}$ satisfies single crossing in $(U_L, U_H, -c)$ which means that $\alpha(U_H, U_L, c)$ is decreasing in U_H, U_L and increasing in c. Moreover, as U_L and U_H are both decreasing in c we can conclude that $\alpha(U_H(c), U_L(c), c)$ is increasing in c, which means that there is $\tilde{c}$ such that for any $c \leq \tilde{c}$ the solution has deterministic monitoring while for any $c > \tilde{c}$ the solution has random monitoring.

Next, we prove that random monitoring dominates deterministic monitoring when k is large enough and when $\bar{a}$ is high or low enough. For this, it is enough to establish that full random monitoring (that is $\hat{\tau} = 0$) dominates fully deterministic as this guarantees that some randomization is going to be used in the optimal policy. Before proving the statements in the proposition, we start proving the following lemma:

Lemma C.1. *For any $\underline{q} \in (0,1)$,*

$$e^{-r\tau^{bind}} > \frac{m^*}{r + m^*}$$

Proof. If we let $\beta \equiv r/(r+\lambda)$, then by replacing τ^{bind} and m^* we can verify that it is enough to show that

$$\underline{q}^\beta - \frac{\underline{q}}{\beta(1-\underline{q}) + \underline{q}} > 0.$$

Consider the function

$$H(q) \equiv \beta q^{\beta-1} + (1-\beta) q^\beta - 1,$$

so we need to show that $H(q) > 0$ for all $q \in (0,1)$. The function H is such $H(0) > 0$ and $H(1) = 0$. Moreover, the derivate of H is given by

$$H'(q) = \beta(\beta-1)q^{\beta-2} + (1-\beta)\beta q^{\beta-1} = -\beta(1-\beta)q^{\beta-2}(1-q) < 0,$$

and so it follows that $H(q) > 0$ for all $q \in (0,1)$. □

Optimality of random monitoring for large k: We compare the payoff of deterministic monitoring with the payoff of full random monitoring (that is $\hat{\tau} = 0$) when k converges to its upper bound, $\lambda/(r+\lambda)$ and show that the difference between the benefit of using random and deterministic monitoring converge to zero while the difference in their cost remains bounded away of zero. For large k, we can restrict attention to monitoring policies in which the IC constraint is binding, and it is enough to compare policies that rely exclusively on deterministic or random monitoring (the argument to rule out policies that alternate between random and deterministic depending on $\theta_{T_{n-1}}$ is analogous).

First, we look at the difference in the cost. The cost of a deterministic policy is

$$C^{\text{det}} = \frac{e^{-r\tau}}{1 - e^{-r\tau}} = \frac{\underline{q}^\beta}{1 - \underline{q}^\beta}$$

while the cost of the random policy is

$$C^{\text{rand}} = \frac{m^*}{r} = \frac{1}{\beta}\frac{\underline{q}}{1-\underline{q}}.$$

The difference in the cost is

$$C^{\text{det}} - C^{\text{rand}} = \frac{\underline{q}^\beta}{1-\underline{q}^\beta} - \frac{1}{\beta}\frac{\underline{q}}{1-\underline{q}} = \frac{1}{\beta}\frac{\beta\underline{q}^\beta - \underline{q} + (1-\beta)\underline{q}^{\beta+1}}{1-\underline{q}-\underline{q}^\beta+\underline{q}^{\beta+1}},$$

and applying L'Hopital's rule twice we find that

$$\begin{aligned}
\lim_{\underline{q}\to 1}\frac{\beta\underline{q}^\beta - \underline{q} + (1-\beta)\underline{q}^{\beta+1}}{1-\underline{q}-\underline{q}^\beta+\underline{q}^{\beta+1}} &= \lim_{\underline{q}\to 1}\frac{\beta^2\underline{q}^{\beta-1} - 1 + (1-\beta)(1+\beta)\underline{q}^\beta}{-1-\beta\underline{q}^{\beta-1}+(\beta+1)\underline{q}^\beta} \\
&= \lim_{\underline{q}\to 1}\frac{\beta(\beta-1)+(1-\beta^2)\underline{q}}{(1-\beta)+(\beta+1)\underline{q}} \\
&= \frac{1-\beta}{2} > 0
\end{aligned}$$

Next, we look at the benefit of monitoring (excluding its cost). First, we compute the benefit of a deterministic policy. The benefit of the deterministic policy, B_θ^{det}, solves the system of equations

$$\begin{aligned}
B_L^{\text{det}} &= \int_0^\tau e^{-rt}u(x_t^L)\mathrm{d}t + e^{-r\tau}(x_\tau^L B_H^{\text{det}} + (1-x_\tau^L)B_L^{\text{det}}) \\
B_H^{\text{det}} &= \int_0^\tau e^{-rt}u(x_t^H)\mathrm{d}t + e^{-r\tau}(x_\tau^H B_H^{\text{det}} + (1-x_\tau^H)B_L^{\text{det}}).
\end{aligned}$$

Solving this system we get that the payoff is given by

$$\begin{aligned}
B_L^{\text{det}} &= \frac{\int_0^\tau e^{-rt}u(x_t^L)\mathrm{d}t}{1-e^{-r\tau}} + \frac{e^{-r\tau}x_\tau^L}{1-e^{-r\tau}(x_\tau^H - x_\tau^L)}\frac{\int_0^\tau e^{-rt}(u(x_t^H)-u(x_t^L))\mathrm{d}t}{1-e^{-r\tau}} \\
B_H^{\text{det}} &= \frac{\int_0^\tau e^{-rt}u(x_t^H)\mathrm{d}t}{1-e^{-r\tau}} - \frac{e^{-r\tau}(1-x_\tau^H)}{1-e^{-r\tau}(x_\tau^H - x_\tau^L)}\frac{\int_0^\tau e^{-rt}(u(x_t^H)-u(x_t^L))\mathrm{d}t}{1-e^{-r\tau}},
\end{aligned}$$

and taking the limit when $\tau \to 0$ (which is equivalent to taking the limit when $k \to \lambda/(r+\lambda)$) we get that

$$\begin{aligned}
B_L^{\text{det}} &\to \frac{1}{r}\left(\frac{r+\lambda(1-\bar{a})}{r+\lambda}u(0) + \frac{\lambda\bar{a}}{r+\lambda}u(1)\right) \\
B_H^{\text{det}} &\to \frac{1}{r}\left(\frac{\lambda(1-\bar{a})}{r+\lambda}u(0) + \frac{r+\lambda\bar{a}}{r+\lambda}u(1)\right)
\end{aligned}$$

On the other hand, the benefit of the random policy is

$$B_L^{\text{rand}} = \int_0^\infty e^{-(r+m^*)t}(u(x_t^L) + m^*(x_t^L B_H^{\text{rand}} + (1-x_t^L)B_L^{\text{rand}}))\text{d}t$$
$$B_H^{\text{rand}} = \int_0^\infty e^{-(r+m^*)t}(u(x_t^H) + m^*(x_t^H B_H^{\text{rand}} + (1-x_t^H)B_L^{\text{rand}}))\text{d}t,$$

where

$$B_L^{\text{rand}} = \int_0^\infty e^{-(r+m^*)t}u(x_t^L)\text{d}t + \frac{m^*}{r+m^*}B_L^{\text{rand}} + \frac{m^*\lambda\bar{a}}{(r+m^*)(r+\lambda+m^*)}(B_H^{\text{rand}} - B_L^{\text{rand}})$$
$$B_H^{\text{rand}} = \int_0^\infty e^{-(r+m^*)t}u(x_t^H)\text{d}t + \frac{m^*}{r+m^*}B_L^{\text{rand}} + \left[\frac{m^*}{r+\lambda+m^*} + \frac{m^*\lambda\bar{a}}{(r+m^*)(r+\lambda+m^*)}\right](B_H^{\text{rand}} - B_L^{\text{rand}})$$

From here we get

$$B_H^{\text{rand}} - B_L^{\text{rand}} = \frac{r+\lambda+m^*}{r+\lambda}\int_0^\infty e^{-(r+m^*)t}(u(x_t^H) - u(x_t^L))\text{d}t$$

So, replacing in the previous equations

$$B_L^{\text{rand}} = \frac{r+m^*}{r}\int_0^\infty e^{-(r+m^*)t}u(x_t^L)\text{d}t + \frac{m^*\lambda\bar{a}}{r(r+\lambda)(r+m^*)}\int_0^\infty (r+m^*)e^{-(r+m^*)t}(u(x_t^H) - u(x_t^L))\text{d}t.$$

We can also write

$$B_H^{\text{rand}} - B_L^{\text{rand}} = \frac{r+\lambda+m^*}{(r+\lambda)(r+m^*)}\int_0^\infty (r+m^*)e^{-(r+m^*)t}(u(x_t^H) - u(x_t^L))\text{d}t$$

From here we get that when $m^* \to \infty$ the benefit converges to

$$B_L^{\text{rand}} \to \frac{1}{r}\left(\frac{r+\lambda(1-\bar{a})}{r+\lambda}u(0) + \frac{\lambda\bar{a}}{r+\lambda}u(1)\right),$$

and

$$B_H^{\text{rand}} - B_L^{\text{rand}} \to \frac{1}{r+\lambda}(u(1) - u(0))$$

so

$$B_H^{\text{rand}} \to \frac{1}{r}\left(\frac{\lambda(1-\bar{a})}{r+\lambda}u(0) + \frac{r+\lambda\bar{a}}{r+\lambda}u(1)\right)$$

Comparing the limit of the deterministic and random policy we verify that both yield the same benefit in the limit of $C^{\text{det}} - C^{\text{rand}}$ is strictly positive, which means that the random policy dominates.

Optimality of random monitoring following $\theta_{T_{n-1}} = H$ for large $\bar{a}$: First, we find an upper bound for payoff of following a deterministic policy

$$\begin{aligned}
\mathcal{G}^{\theta}_{\text{det}}(U) &= \int_0^{\tau} e^{-rt} u\left(x^{\theta}_t\right) \mathrm{d}t + e^{-r\tau}\left[U_L - c + \overline{a}\Delta U + (\theta - \overline{a})\, e^{-\lambda\tau}\Delta U\right] \\
&< \frac{u(1)}{r}(1 - e^{-r\tau}) + e^{-r\tau}(U_H - c) \\
&\leq \frac{u(1)}{r}(1 - e^{-r\tau^{\text{bin}}}) + e^{-r\tau^{\text{bin}}}(U_H - c) \\
&= \frac{u(1)}{r}(1 - \underline{q}^{\frac{r}{r+\lambda}}) + \underline{q}^{\frac{r}{r+\lambda}}(U_H - c)
\end{aligned}$$

Next, we find a lower bound for the payoff of following a random policy

$$\begin{aligned}
\mathcal{G}^{\theta}_{\text{rand}}(U) &= \int_0^{\infty} e^{-(r+m^*)t}\left[u\left(x^{\theta}_t\right) + m^*\mathcal{M}\left(U, x^{\theta}_t\right)\right]\mathrm{d}t \\
&> \int_0^{\infty} e^{-(r+m^*)t}\mathrm{d}t\left[u(\bar{a}) + m^*(\bar{a}U_H + (1-\bar{a})\,U_L - c)\right] \\
&= \frac{u(\bar{a})}{r+m^*} + \frac{m^*(\bar{a}U_H + (1-\bar{a})\,U_L - c)}{r+m^*}
\end{aligned}$$

Finally, we show that if $\bar{a}$ is large enough, then the upper bound for $\mathcal{G}^{\theta}_{\text{det}}(U)$ is below the lower bound for $\mathcal{G}^{\theta}_{\text{rand}}$. This requires that for any U we have

$$\frac{u(1)}{r}(1 - \underline{q}^{\frac{r}{r+\lambda}}) + \underline{q}^{\frac{r}{r+\lambda}}(U_H - c) \leq \frac{u(\bar{a})}{r+m^*} + \frac{m^*(\bar{a}U_H + (1-\bar{a})\,U_L)}{r+m^*}$$

Following the proof in Lemma C.1, we let $\beta \equiv \frac{r}{r+\lambda}$ so we can write

$$\begin{aligned}
\frac{u(\bar{a})}{r+m^*} + \frac{m^*(\bar{a}U_H + (1-\bar{a})\,U_L)}{r+m^*} &= \frac{u(\bar{a})}{r}(1 - \underline{q}^{\beta}) + u(\bar{a})\left(\frac{\underline{q}^{\beta} - 1}{r} + \frac{1}{r+m^*}\right) \\
&\quad + q^{\beta}(\bar{a}U_H + (1-\bar{a})\,U_L - c) \\
&\quad + \left(\frac{m^*}{r+m^*} - q^{\beta}\right)(\bar{a}U_H + (1-\bar{a})\,U_L - c)
\end{aligned}$$

Letting $\Delta U \equiv U_H - U_L$, we write our required inequality as

$$\left(\frac{u(1)}{r} - \frac{u(\bar{a})}{r}\right)(1 - \underline{q}^{\beta}) \leq \frac{u(\bar{a})}{r}\left(\underline{q}^{\beta} - \frac{m^*}{r+m^*}\right) + \left(\frac{m^*}{r+m^*} - q^{\beta}\right)(U_H - c) - \frac{m^*}{r+m^*}(1-\bar{a})\Delta U,$$

and after replacing m^* we reduce it to

$$\left(\frac{u(1)}{r} - \frac{u(\bar{a})}{r}\right)(1 - \underline{q}^{\beta}) \leq \left(\frac{u(\bar{a})}{r} + c - U_H\right)\left(\underline{q}^{\beta} - \frac{\underline{q}}{\beta(1-\underline{q}) + \underline{q}}\right) - \frac{\underline{q}(1-\bar{a})\Delta U}{\beta(1-\underline{q}) + \underline{q}}$$

Clearly, it must be the case that $\frac{u(1)}{r} > U_H$, which means that

$$\lim_{\bar{a}\to 1}\left(\frac{u(1)}{r}-\frac{u(\bar{a})}{r}\right)(1-\underline{q}^{\beta}) = 0$$
$$< \left(\frac{u(1)}{r}+c-U_H\right)\left(\underline{q}^{\beta}-\frac{\underline{q}}{\beta(1-\underline{q})+\underline{q}}\right)$$
$$= \lim_{\bar{a}\to 1}\left\{\left(\frac{u(\bar{a})}{r}+c-U_H\right)\left(\underline{q}^{\beta}-\frac{\underline{q}}{\beta(1-\underline{q})+\underline{q}}\right)-\frac{\underline{q}(1-\bar{a})\Delta U}{\beta(1-\underline{q})+\underline{q}}\right\},$$

and so there is $\epsilon > 0$ such that for all $\bar{a} \in (1-\epsilon, 1)$ we have that $\mathcal{G}^{\theta}_{\text{det}}(U) < \mathcal{G}^{\theta}_{\text{rand}}(U)$

Optimality of random monitoring following $\theta_{T_{n-1}} = L$ for small $\bar{a}$: The proof follows a similar argument as the one for large $\bar{a}$. The payoff of the deterministic policy satisfies the inequality

$$\begin{aligned}\mathcal{G}^{\theta}_{\text{det}}(U) &< \int_0^{\tau} e^{-rt}u(\overline{a})\,\text{d}t + e^{-r\tau}[U_L - c + \overline{a}\Delta U + (\theta-\overline{a})e^{-\lambda\tau}\Delta U] \\ &= \frac{u(\overline{a})}{r}\left(1-e^{-r\tau}\right) + e^{-r\tau}[U_L - c + \overline{a}\Delta U\left[1-e^{-\lambda\tau}\right]]\end{aligned}$$

Replacing τ_{bind} and taking the limit when $\bar{a}$ goes to zero we find

$$\lim_{\overline{a}\to 0}\mathcal{G}^{\theta}_{\text{det}}(U) < \frac{u(0)}{r}\left(1-e^{-r\tau_{\text{bind}}}\right) + e^{-r\tau_{\text{bind}}}\lim_{\overline{a}\to 0}[U_L - c]$$

Similarly, the payoff of the random policy satisfies

$$\begin{aligned}\mathcal{G}^{\theta}_{\text{rand}}(U) &= \int_0^{\infty} e^{-(r+m^*)t}\left[u\left(x^{\theta}_t\right)+m^*\mathcal{M}\left(U, x^{\theta}_t\right)\right]\text{d}t \\ &= \frac{1}{r+m^*}\int_0^{\infty}(r+m^*)e^{-(r+m^*)t}\left[u\left(x^{\theta}_t\right)+m^*\mathcal{M}\left(U, x^{\theta}_t\right)\right]\text{d}t \\ &> \frac{[u\left(\frac{a\lambda}{r+m+\lambda}\right)+\frac{a\lambda}{r+m+\lambda}m^*U_H + m^*(1-\frac{a\lambda}{r+m+\lambda})U_L - m^*c]}{r+m^*},\end{aligned}$$

and so the limit when $\bar{a}$ goes to zero is

$$\lim_{\overline{a}\to 0}\mathcal{G}^{\theta}_{\text{rand}}(U) > \frac{r\frac{u(0)}{r}+m^*\lim_{\overline{a}\to 0}(U_L-c)}{r+m^*}$$

In the limit, it must be the case that $\frac{u(0)}{r} \geq \lim_{\overline{a}\to 0}(U_L - c)$: If fact

$$\lim_{\overline{a}\to 0}U_L < \lim_{\overline{a}\to 0}E\left[\int_0^{\infty}e^{-rt}u(\theta_t)\,dt|\theta_0 = L\right],$$

and by dominated convergence

$$\begin{aligned}\lim_{\bar{a}\to 0} E\left[\int_0^\infty e^{-rt} u\left(\theta_t\right) \mathrm{d}t | L\right] &= \int_0^\infty e^{-rt} \lim_{\bar{a}\to 0} E\left[u\left(\theta_t\right)|\theta_0 = L\right] \mathrm{d}t \\ &= \frac{u\left(0\right)}{r}.\end{aligned}$$

From Lemma C.1 we have that $e^{-r\tau_{\text{bind}}} > \frac{m^*}{r+m^*}$, and so it follows that

$$\lim_{\bar{a}\to 0} \mathcal{G}^{\theta}_{\text{rand}}\left(U\right) - \lim_{\bar{a}\to 0} \mathcal{G}^{\theta}_{\text{det}}\left(U\right) > 0.$$

This means that there is $\epsilon > 0$ such that the random policy dominates the deterministic policy for any $\bar{a} \in (0, \epsilon)$

Optimality of monitoring at constant rate m^* when $\lambda \to \infty$. We verify that when $\lambda \to \infty$, the optimal policy is full random monitoring. With full random monitoring, we have that

$$\begin{aligned}U_\theta &= \int_0^\infty e^{-(r+m^*)\tau} \left(u(x^\theta_\tau) + m^* \left(U_L + x^\theta_\tau (U_H - U_L) - c\right)\right) \mathrm{d}\tau \\ &= \frac{m^*}{r+m^*} \left(U_L + \bar{a}(U_H - U_L) - c\right) + \frac{m^*}{r+m^*+\lambda}(\theta - \bar{a})(U_H - U_L) \\ &\quad + \int_0^\infty e^{-(r+m^*)\tau} u\left(\theta e^{-\lambda\tau} + \bar{a}(1-e^{-\lambda\tau})\right) \mathrm{d}\tau.\end{aligned}$$

From here we get that

$$U_H - U_L = \frac{m^*}{r+m^*+\lambda}(U_H - U_L) + \int_0^\infty e^{-(r+m^*)\tau} \left[u\left(e^{-\lambda\tau} + \bar{a}(1-e^{-\lambda\tau})\right) - u\left(\bar{a}(1-e^{-\lambda\tau})\right)\right] \mathrm{d}\tau$$

Substituting

$$m^* = (r+\lambda)\frac{\underline{q}}{1-\underline{q}},$$

and solving for $U_H - U_L$ we get

$$U_H - U_L = \frac{1}{1-\underline{q}} \int_0^\infty e^{-(r+m^*)\tau} \left[u\left(e^{-\lambda\tau} + \bar{a}(1-e^{-\lambda\tau})\right) - u\left(\bar{a}(1-e^{-\lambda\tau})\right)\right] \mathrm{d}\tau$$

From here we get that $\lim_{\lambda\to\infty}(U_H - U_L) = 0$ and $\lim_{\lambda\to\infty} m^*(U_H - U_L) = 0$. We also have that

$$\begin{aligned}r(U_L - c) &= \bar{a}m^*(U_H - U_L) - (r+m^*)c + \frac{r+m^*}{r+m^*+\lambda}(\theta - \bar{a})m^*(U_H - U_L) \\ &\quad + \int_0^\infty (r+m^*)e^{-(r+m^*)\tau} u(x^\theta_\tau) \mathrm{d}\tau \\ &\to_{\lambda\to\infty, c\to 0, \lambda c<\infty} = u(x^\theta_0) - \frac{k}{1-k} c_\lambda,\end{aligned}$$

where $c_\lambda \equiv \lim_{\lambda\to\infty,c\to 0}\lambda c < \infty$. Having solved for the limits of $U_H - U_L$ and $U_L - c$, the next step is to verify that in the limit

$$h_\theta(0) < \int_0^\infty \rho e^{-\rho\tau} h_\theta(\tau)\mathrm{d}\tau,$$

which, by Proposition 2, would provide a verification that constant monitoring at rate m^* is optimal. Substituting the definition of h_θ and $\rho = r + \lambda + m^*$ we get that

$$\begin{aligned} h_\theta(0) - \int_0^\infty \rho e^{-\rho\tau} h_\theta(\tau)\mathrm{d}\tau = u(x_0^\theta) - \left(1 + \frac{\lambda(1-\underline{q})}{r+\lambda\underline{q}}\right)\int_0^\infty (r+m^*)e^{-(r+m^*)\tau}u(x_\tau^\theta)\mathrm{d}\tau \\ + \frac{\lambda(1-\underline{q})}{r+\lambda\underline{q}} r\left(U_L + \bar{a}(U_H - U_L) - c\right) \end{aligned}$$

Taking the limit we get

$$\lim_{\lambda\to\infty,c\to 0,c_\lambda<\infty}\left(h_\theta(0) - \int_0^\infty \rho e^{-\rho\tau}h_\theta(\tau)\mathrm{d}\tau\right) = -c_\lambda < 0.$$

C.2 Proof of Proposition 6

Proof. For the first part, notice that the existence of $\tilde{c}^\dagger$ follows directly from Proposition 2a. Next, let's define

$$\begin{aligned} H^{\text{det}}(\bar\tau) &\equiv \frac{\int_0^{\bar\tau} e^{-r\tau}\gamma\Sigma_\tau \mathrm{d}\tau + e^{-r\bar\tau}c}{1-e^{-r\bar\tau}} \\ H^{\text{rand}}(\hat\tau) &\equiv \frac{\int_0^{\hat\tau} e^{-r\tau}\gamma\Sigma_\tau \mathrm{d}\tau + e^{-r\hat\tau}\left(\frac{1-e^{(r+\lambda)\hat\tau}\underline{q}}{1-\underline{q}}\right)\int_{\hat\tau}^\infty e^{-(r+m^*)(\tau-\hat\tau)}\gamma\Sigma_\tau\mathrm{d}\tau + \delta(\hat\tau)c}{1-\delta(\hat\tau)} \end{aligned}$$

We have that

$$\begin{aligned} H^{\text{det}}_{c\bar\tau} &= -\frac{re^{-r\bar\tau}}{(1-e^{-r\bar\tau})^2} < 0 \\ H^{\text{rand}}_{c\hat\tau} &= \frac{\delta'(\hat\tau)}{(1-\delta(\hat\tau))^2} > 0, \end{aligned}$$

which means that $\bar\tau$ is increasing in c, and $\hat\tau^*$ is decreasing in c and p^* is increasing in c.

Next, let's consider the comparative statics with respect to k. First, notice that $H^{\text{det}}(\bar\tau, c)$ is independent of k and that the cost of effort becomes relevant only once the incentive compatibility constraint is binding. Next, we consider the maximization of $H^{\text{rand}}(\hat\tau)$. Because k enters into the maximization problem only through $\underline{q}$ it is enough to show that $\hat\tau$ is decreasing in $\underline{q}$. After some lengthy computations, we have that $H_{\hat\tau} = 0$ if and only if

$$\tilde{g}(\hat\tau, \underline{q}) \equiv r(r+\lambda(2-\underline{q}))\left(2\tilde{c}\lambda(r+2\lambda)-1\right) + 2\underline{q}r(r+2\lambda)e^{-\lambda\hat\tau} - (r+2\lambda)(r+\lambda\underline{q})e^{-2\lambda\hat\tau} + 2\lambda^2\underline{q}e^{-(r+2\lambda)\hat\tau} = 0$$

Let $z \equiv e^{-(r+2\lambda)\hat{\tau}}$ and write

$$g(z,\underline{q}) \equiv \tilde{g}(-\log(z)/(r+2\lambda),\underline{q}) = r(r+\lambda(2-\underline{q}))\left(2\tilde{c}\lambda(r+2\lambda)-1\right) \\ + 2\underline{q}r(r+2\lambda)z^{\frac{\lambda}{r+2\lambda}} - (r+2\lambda)(r+\lambda\underline{q})z^{\frac{2\lambda}{r+2\lambda}} + 2\lambda^2\underline{q}z. \quad \text{(C.1)}$$

The incentive compatibility constraint requires that $\hat{\tau} \leq \tau^{\text{bind}}$, which means that

$$z \geq \underline{q}^{\frac{r+2\lambda}{r+\lambda}}.$$

It follows from here that

$$g_z(z,\underline{q}) = 2\underline{q}r\lambda z^{-\frac{r+\lambda}{r+2\lambda}} - 2\lambda(r+\lambda\underline{q})z^{-\frac{r}{r+2\lambda}} + 2\lambda^2\underline{q} \\ \leq 2\lambda\left(r+\lambda\underline{q}\right)\left(1-z^{-\frac{r}{r+2\lambda}}\right) \leq 0.$$

so we only need to verify that $g_{\underline{q}}(z,\underline{q}) > 0$. Notice that $g(z,\underline{q})$ is linear in $\underline{q}$, so we can write $g(z,\underline{q}) = g_0(z)+g_1(z)\underline{q}$. Hence, if $g(\hat{z},\underline{q}) = 0$, then it must be the case that $g_1(\hat{z})\underline{q} = -g_0(\hat{z})$, which means that it is enough to show that $g_0(\hat{z}) < 0$ evaluated at the solution. Substituting in equation (C.1) we have that

$$g_0(z) = r(r+2\lambda)\left[2\tilde{c}\lambda(r+2\lambda) - \left(1+z^{\frac{2\lambda}{r+2\lambda}}\right)\right] < 0,$$

where the last inequality follows from the condition in the Proposition

$$\tilde{c} \leq \frac{1}{2\lambda(r+2\lambda)}.$$

Finally, we verify that the optimal policy in the i.i.d. limit is random. From the first order condition for $\hat{\tau}$, the optimal policy is random if $g(1,\underline{q}) \geq 0$, which reduces to

$$\lambda\tilde{c} \geq \frac{1-\underline{q}}{r+\lambda(2-\underline{q})}.$$

When this condition holds, the optimal policy is monitoring with a constant hazard rate starting at time zero. The left-hand side converges to zero as $\lambda \to \infty$, so indeed, random monitoring is optimal. $\square$

D Proofs of Analysis Using Optimal Control

D.1 Existence: Proof of Lemma 6

Proof. The first step in the proof is to show that the operator $\mathcal{G}^\theta$ has a unique fixed point. Let's denote the vector of expected payoffs by $U \equiv (U_L, U_H)$. We have that $U^{\max} = (u(1) - k\bar{a})/r < \infty$ is an upper bound for the principal payoff. The monitoring policy $m_t = 0$, and $\bar{\tau}$ solving $e^{-(r+\lambda)\bar{\tau}} = \underline{q}$ provides a lower bound $U_\theta^{\min} > -\infty$. We consider the rectangle $R = [U_L^{\min}, U^{\max}] \times [U_H^{\min}, U^{\max}]$. Let $\mathscr{G}_\epsilon^\theta$ be the Bellman operator with the extra constraint that $E(e^{-rT}) = \int_0^\infty e^{-rt} dF(t) \leq e^{-r\epsilon}$. For any bounded functions f, g we have that $|\sup f - \sup g| \leq \sup |f - g|$, and so because the function $\mathscr{G}_\epsilon = (\mathscr{G}_\epsilon^L, \mathscr{G}_\epsilon^H)$ is bounded in R, we have that

$$\|\mathscr{G}_\epsilon U^0 - \mathscr{G}_\epsilon U^1\| \leq e^{-r\epsilon} \|U^0 - U^1\|.$$

Hence, by the Contraction Mapping Theorem there is a unique fixed-point $\mathscr{G}_\epsilon U_\epsilon = U_\epsilon$. For any sequence $\epsilon_k \downarrow 0$ we have that the sequence U_{ϵ_k} is increasing and bounded above by $U^{\max}$: Accordingly, U_{ϵ_k} converges to some limit $\overline{U}$, and because $\mathscr{G}_\epsilon$ is lower semicontinuous as a function of ϵ (Aliprantis and Border, 2006, Lemma 17.29) we also have that

$$\lim_{\epsilon_k \downarrow 0} \mathscr{G}_{\epsilon_k} U_{\epsilon_k} \geq \mathscr{G} \overline{U}.$$

On the other hand, $\mathscr{G}_\epsilon$ is increasing in U, decreasing in ϵ and U_{ϵ_k} is an increasing sequence so

$$\lim_{\epsilon_k \downarrow 0} \mathscr{G}_{\epsilon_k} U_{\epsilon_k} \leq \mathscr{G} \overline{U}.$$

Accordingly, $\lim_{\epsilon_k \downarrow 0} \mathscr{G}_{\epsilon_k} U_{\epsilon_k} = \mathscr{G} \overline{U}$ and we conclude that

$$\overline{U} = \lim_{\epsilon_k \downarrow 0} U_{\epsilon_k} = \lim_{\epsilon_k \downarrow 0} \mathscr{G}_{\epsilon_k} U_{\epsilon_k} = \mathscr{G} \overline{U}.$$

The next step is to show that a solution to the maximization problem exists. To prove existence, we consider the space of probability measures over $\mathbb{R}_+ \cup \{\infty\}$, which we denote by $\mathcal{P}$, endowed with the weak* topology. The extended reals $\mathbb{R}_+ \cup \{\infty\}$ are a metrizable compact space so by Theorem 15.11 in Aliprantis and Border (2006) the space $\mathcal{P}$ is compact in the weak* topology. The incentive compatibility constraint can be written $\int_\tau^\infty e^{-(r+\lambda)(s-\tau)} \mathrm{d}F(s) \geq \underline{q}(1 - F(\tau-))$ for all $\tau \in \mathbb{R}_+ \cup \{\infty\}$ which means that the set of incentive compatible monitoring policies is a closed subset of $\mathcal{P}$, and so a compact set. Finally, the objective function is a bounded linear functional on $C(\mathbb{R}_+ \cup \{\infty\})$ so it is continuous in the weak* topology, and thus is maximized by some incentive compatible policy F^*. ∎

D.2 Proof of Theorem 1 Using Optimal Control

Proof of Lemma 8

Proof. At any point of continuity we have that

$$\mathrm{d}\nu_\tau = -\lambda\nu_\tau \mathrm{d}\tau - \mathrm{d}\Phi_\tau. \tag{D.1}$$

We also have the optimality conditions

$$S(\tau) \leq 0 \tag{D.2a}$$

$$M_\tau = \int_0^\tau \mathbf{1}_{\{S(u)=0\}} \mathrm{d}M_u. \tag{D.2b}$$

Condition (D.2a) corresponds to

$$S(\tau) = \mathcal{M}(\mathbf{U}, x_\tau^\theta) - U_\tau - (1-q_\tau)\nu_\tau \leq 0.$$

Differentiation $S(\tau)$ we find

$$\begin{aligned}
\mathrm{d}S(\tau) =& \dot{x}_\tau^\theta (U_H - U_L)\mathrm{d}\tau - \mathrm{d}U_\tau + \nu_\tau \mathrm{d}q_\tau - (1-q_\tau)\mathrm{d}\nu_\tau \\
=& \dot{x}_\tau^\theta (U_H - U_L)\mathrm{d}\tau - \left(rU_\tau - u(x_\tau^\theta)\right)\mathrm{d}\tau - \left(U_\tau - \mathcal{M}(\mathbf{U}, x_\tau^\theta)\right)\mathrm{d}M_\tau^c \\
&+ \nu_\tau \left((r+\lambda)q_\tau \mathrm{d}t - (1-q_\tau)\mathrm{d}M^c(\tau)\right) + (1-q_\tau)\left(\lambda\nu_\tau \mathrm{d}\tau + \mathrm{d}\Phi_\tau\right) \\
=& \left(\dot{x}_\tau^\theta (U_H - U_L) + u(x_\tau^\theta) - rU_\tau + \nu_\tau(rq_\tau + \lambda)\right)\mathrm{d}t + (1-q_\tau)\mathrm{d}\Phi_\tau + S(\tau)\mathrm{d}M_\tau^c
\end{aligned}$$

The optimality condition (D.2b) implies that $S(\tau)\mathrm{d}M_\tau = 0$. Thus we can write the evolution of $S(\tau)$ as

$$\mathrm{d}S(\tau) = \left(\dot{x}_\tau^\theta (U_H - U_L) + u(x_\tau^\theta) - rU_\tau + \nu_\tau(rq_\tau + \lambda)\right)\mathrm{d}t + (1-q_\tau)\mathrm{d}\Phi_\tau. \tag{D.3}$$

Whenever $q_\tau > \underline{q}$ we have that $\mathrm{d}\Phi_\tau = 0$, which means that $S(\tau)$ is absolutely continuous in any interval (τ', τ'') with $q_\tau > \underline{q}$ (notice that q_τ is continuous between jumps so wlog we can assume that if $q_{\tilde{\tau}} > \underline{q}$ at some time $\tilde{\tau}$ between jumps then there is neighborhood of $\tilde{\tau}$ such that $q_\tau > \underline{q}$) . Note as well that $S(\tau)\mathrm{d}M_\tau^c = 0$ implies that we can write

$$\mathrm{d}U_\tau = \left(rU_\tau - u(x_\tau^\theta)\right)\mathrm{d}\tau - (1-q_\tau)\nu_\tau \mathrm{d}M_\tau^c \tag{D.4}$$

Let $\dot{S}(\tau)$ denote the drift of $S(\tau)$, which is given by

$$\dot{S}(\tau) \equiv \dot{x}_\tau^\theta (U_H - U_L) + u(x_\tau^\theta) - rU_\tau + \nu_\tau(rq_\tau + \lambda). \tag{D.5}$$

Differentiating $\dot{S}(\tau)$ we find

$$\mathrm{d}\dot{S}(\tau) = \left(\ddot{x}_\tau^\theta (U_H - U_L) + u'(x_\tau^\theta)\dot{x}_\tau^\theta\right)\mathrm{d}\tau - r\mathrm{d}U_\tau + (rq_\tau + \lambda)\mathrm{d}\nu_\tau + r\nu_\tau \mathrm{d}q_\tau \tag{D.6}$$

Replacing equations (D.1) and (D.4), and the equation for $\mathrm{d}q_\tau$ in (D.6) we find that

$$\mathrm{d}\dot{S}(\tau) = \Big(r^{-1}\ddot{x}^\theta_\tau(U_H - U_L) + r^{-1}u'(x^\theta_\tau)\dot{x}^\theta_\tau - rU_\tau + u(x^\theta_\tau) + r^{-1}(r^2 q_\tau - \lambda^2)\nu_\tau\Big)\,\mathrm{d}\tau. \tag{D.7}$$

The support of M^c_τ is $A \equiv \{\tau : S(\tau) = 0\}$, which correspond to the set of maximizers of $S(\tau)$. Accordingly, for any time $\tau \in A$, we have that $S(\tau) = \dot{S}(\tau) = 0$ and $\ddot{S}(\tau) \leq 0$. Suppose that there is $\tilde{\tau}$ such that $S(\tau) = 0$, $\dot{S}(\tilde{\tau}) = 0$ and $\ddot{S}(\tilde{\tau}) = 0$, and replacing $S(\tilde{\tau}) = 0$ and $\ddot{x}^\theta_\tau = -\lambda\dot{x}^\theta_\tau$ in (D.7), then we get that it must be the case that

$$\nu_{\tilde{\tau}} = \frac{\dot{x}^\theta_{\tilde{\tau}}}{\lambda(r+\lambda)}\Big(u'(x^\theta_{\tilde{\tau}}) - (r+\lambda)(U_H - U_L)\Big) \tag{D.8}$$

Let's define

$$z_\tau \equiv \frac{\dot{x}^\theta_\tau}{\lambda(r+\lambda)}\Big(u'(x^\theta_\tau) - (r+\lambda)(U_H - U_L)\Big).$$

Differentiating z_τ we get

$$\begin{aligned}
\mathrm{d}z_\tau &= \left(\frac{\ddot{x}^\theta_\tau}{\lambda(r+\lambda)}\Big(u'(x^\theta_\tau) - (r+\lambda)(U_H - U_L)\Big) + \frac{(\dot{x}^\theta_\tau)^2}{\lambda(r+\lambda)}u''(x^\theta_\tau)\right)\mathrm{d}t \\
&= \left(\frac{\ddot{x}^\theta_\tau}{\dot{x}^\theta_\tau}z_\tau + \frac{(\dot{x}^\theta_\tau)^2}{\lambda(r+\lambda)}u''(x^\theta_\tau)\right)\mathrm{d}\tau \\
&= \left(-\lambda z_\tau + \frac{(\dot{x}^\theta_\tau)^2}{\lambda(r+\lambda)}u''(x^\theta_\tau)\right)\mathrm{d}\tau
\end{aligned}$$

On the other hand, whenever $q_\tau > \underline{q}$ we have that $\mathrm{d}\Phi_\tau = 0$ so

$$\mathrm{d}\nu_\tau = -\lambda\nu_\tau\mathrm{d}\tau.$$

Accordingly

$$\mathrm{d}(\nu_\tau - z_\tau) = -\lambda(\nu_\tau - z_\tau)\mathrm{d}\tau - \frac{(\dot{x}^\theta_\tau)^2}{\lambda(r+\lambda)}u''(x^\theta_\tau)\mathrm{d}\tau,$$

so for any $\tau > \tilde{\tau}$

$$\nu_\tau - z_\tau = \int_{\tilde{\tau}}^{\tau} e^{-\lambda(\tau-s)}\frac{(\dot{x}^\theta_s)^2}{\lambda(r+\lambda)}u''(x^\theta_s)\mathrm{d}s > 0.$$

This means that there is at most one $\tilde{\tau} \in A$ satisfying equation (D.8), which means that there is at most one $\tilde{\tau} \in A$ such that $\ddot{S}(\tilde{\tau}) = 0$, and any other $\tau \in A$ satisfies $\ddot{S}(\tau) < 0$. This means that all, but at most one, $\tau \in A$, are isolated points. And, by Theorem 7.14.23 in (Bogachev, 2007), the only atomless measure in A is the trivial zero measure, which means that $M^c_\tau - M^c_{\tau'} = 0$ for all $\tau \in [\tau', \tau'')$

□

Proof of Lemma 9

Proof. The first step is to verify that $S(\tau)$ is continuous at any atom τ_k. We have that

$$S(\tau_k-) = \mathcal{M}(\mathbf{U}, x^\theta_{\tau_k}) - U_{\tau_k-} - \nu_{\tau_k-}(1-q_{\tau_k-})$$

Using the fact that ν_τ satisfies

$$\nu_\tau = \nu_0 - \int_0^\tau \lambda\nu_s \mathrm{d}s - \Phi_\tau, \tag{D.9}$$

we find that

$$\begin{aligned} S(\tau_k) &= \mathcal{M}(\mathbf{U}, x^\theta_{\tau_k}) - U_{\tau_k} - (\nu_{\tau_k-} - \Delta\Phi_{\tau_k})(1-q_{\tau_k}) \\ &= e^{\Delta M^d_{\tau_k}}\left(\mathcal{M}(\mathbf{U}, x^\theta_{\tau_k}) - U_{\tau_k-} - (\nu_{\tau_k-} - \Delta\Phi_{\tau_k})(1-q_{\tau_k-})\right) \\ &= e^{\Delta M^d_{\tau_k}}\left(S(\tau_k-) + \Delta\Phi_{\tau_k}(1-q_{\tau_k})\right) \\ &= e^{\Delta M^d_{\tau_k}}\Delta\Phi_{\tau_k}(1-q_{\tau_k}), \end{aligned}$$

where we have used that $S(\tau_k) = 0$ at any atom τ_k. Because $S(\tau_k) \leq 0$ and Φ_τ is non-decreasing, we can conclude that $\Delta\Phi_{\tau_k} = 0$, which means that ν_τ is continuous at τ_k. Hence, at any jump atom, the following necessary condition must hold

$$r\mathcal{M}(\mathbf{U}, x^\theta_{\tau_k}) = u(x^\theta_{\tau_k}) + \dot{x}^\theta_{\tau_k}(U_H - U_L) + (r+\lambda)\nu_{\tau_k-} \tag{D.10}$$

The objective now is to show that equation (D.10) cannot be satisfied at more than one point. Let's define

$$G(\tau) \equiv u(x^\theta_\tau) + \dot{x}^\theta_\tau(U_H - U_L) + (r+\lambda)\nu_\tau - r\mathcal{M}(\mathbf{U}, x^\theta_\tau)$$

We have from equation (D.3) that

$$\mathrm{d}S(\tau) = \dot{S}(\tau)\mathrm{d}\tau + (1-q_\tau)\mathrm{d}\Phi_\tau,$$

where we notice that

$$\dot{S}(\tau) = u(x^\theta_\tau) + \dot{x}^\theta_\tau(U_H - U_L) - rU_\tau + \nu_\tau(rq_\tau + \lambda) = rS(\tau) + G(\tau). \tag{D.11}$$

Accordingly, for any atom τ_k, the following conditions must be satisfied

$$\begin{aligned} S(\tau_k-) &= 0 \\ G(\tau_k-) &= 0 \\ \dot{S}(\tau_k-) &= 0. \end{aligned}$$

As $\Delta\Phi_{\tau_k} = 0$, then both $G(\tau)$ and $S(\tau)$ are continuous at the atom τ_k, and $G(\tau_k) = S(\tau_k) = 0$, which means that $\dot{S}(\tau_k) = 0$. Moreover, because τ_k is a local maximum of $S(\tau)$, and $S(\tau)$ is differentiable at τ_k, it follows that that $\ddot{S}(\tau_k-) \leq 0$. Equation (D.11) then implies that $\dot{G}(\tau_k-) \leq 0$. Differentiating $G(\tau)$, we find that

$$\mathrm{d}G(\tau) = \Big(u'(x^\theta_\tau)\dot{x}^\theta_\tau - (r+\lambda)\big(\dot{x}^\theta_\tau(U_H - U_L) + \lambda\nu_\tau\big)\Big)\,\mathrm{d}\tau - (r+\lambda)\mathrm{d}\Phi_\tau$$

Let's $J(\tau)$ be given by

$$J(\tau) \equiv u'(x^\theta_\tau)\dot{x}^\theta_\tau - (r+\lambda)\dot{x}^\theta_\tau(U_H - U_L) - (r+\lambda)\lambda\nu_\tau. \tag{D.12}$$

Notice that whenever the IC constraint is slack we have $\dot{G}(\tau) = J(\tau)$, so in particular $\dot{G}(\tau_k-) = J(\tau_k-)$ for any atom τ_k. Next, if we differentiate equation (D.12) we get

$$\begin{aligned}\mathrm{d}J(\tau) &= \Big(u''(x^\theta_\tau)(\dot{x}^\theta_\tau)^2 - \lambda u'(x^\theta_\tau)\dot{x}^\theta_\tau - (r+\lambda)(-\lambda\dot{x}^\theta_\tau)(U_H - U_L)\Big)\,\mathrm{d}\tau - \lambda(r+\lambda)\mathrm{d}\nu_\tau \\ &= \Big(u''(x^\theta_\tau)(\dot{x}^\theta_\tau)^2 - \lambda u'(x^\theta_\tau)\dot{x}^\theta_\tau + (r+\lambda)\lambda\dot{x}^\theta_\tau(U_H - U_L)\Big)\,\mathrm{d}\tau + \lambda(r+\lambda)(\lambda\nu_\tau\mathrm{d}\tau + \mathrm{d}\Phi_\tau) \\ &= \Big(u''(x^\theta_\tau)(\dot{x}^\theta_\tau)^2 - \lambda u'(x^\theta_\tau)\dot{x}^\theta_\tau + (r+\lambda)\lambda(\dot{x}^\theta_\tau(U_H - U_L) + \lambda\nu_\tau)\Big)\,\mathrm{d}\tau + \lambda(r+\lambda)\mathrm{d}\Phi_\tau,\end{aligned}$$

which can be rewritten as

$$\mathrm{d}J(\tau) = \Big(u''(x^\theta_\tau)(\dot{x}^\theta_\tau)^2\Big)\,\mathrm{d}\tau - \lambda J(\tau)\mathrm{d}\tau + \lambda(r+\lambda)d\Phi_\tau.$$

Thus, for any $\tau \in [\tau_k, \tau_{k+1})$ we have

$$\begin{aligned}J(\tau) &= -\int_\tau^{\tau_{k+1}} e^{\lambda(s-\tau)}\,\Big((\dot{x}^\theta_s)^2 u''(x^\theta_s)\mathrm{d}s + \lambda(r+\lambda)\mathrm{d}\Phi_s\Big) + e^{\lambda(\tau_{k+1}-\tau)}J(\tau_{k+1}-) \\ &= -\int_\tau^{\tau_{k+1}} e^{\lambda(s-\tau)}\,\Big((\dot{x}^\theta_s)^2 u''(x^\theta_s)\mathrm{d}s + \lambda(r+\lambda)\mathrm{d}\Phi_s\Big) + e^{\lambda(\tau_{k+1}-\tau)}\dot{G}(\tau_{k+1}-) < 0,\end{aligned}$$

where we have used the fact that $J(\tau_{k+1}-) = \dot{G}(\tau_{k+1}-) \leq 0$. But then,

$$\mathrm{d}G(\tau) = J(\tau)\mathrm{d}\tau - (r+\lambda)\mathrm{d}\Phi_\tau < 0$$

for all $\tau \in (\tau_k, \tau_{k+1})$ which contradicts the requirement that $G(\tau_{k+1}-) = 0$.

□

Proof of Theorem 1

Proof. Lemma 8 implies that, in the absence of an atom, q_τ is increasing if $q_\tau > \underline{q}$ because q_τ increases whenever $\mathrm{d}M^{c*}_\tau = 0$. Hence, because there is at most one atom, this means that either there is monitoring with probability one at the atom, or the incentive compatibility constraint is binding thereafter. If this were not the case, q_τ would eventually reach one, which would require a second atom and contradict lemma 9. Thus lemmas 8 and 9 imply that the optimal monitoring policy takes the following form:

1. There is $\tilde{\tau}$ such that for any $\tau \in [0, \tilde{\tau})$ we have $q_\tau = \underline{q}$.

2. There is $\hat{\tau}$ such that for any $\tau \in [\tilde{\tau}, \hat{\tau})$ there is no monitoring and $q_\tau > \underline{q}$.

3. There is an atom at time $\hat{\tau}$. If the probability of monitoring at the atom is less than one, then there is a constant rate of monitoring after $\hat{\tau}$.

Thus, the problem of solving for the optimal policy is reduced to finding $\tilde{\tau}$ and $\hat{\tau}$. The last step of the proof shows that $\tilde{\tau}$ is either zero or infinity. The intuition is the following. Analogous to standard contracting models, equation (D.1) works as a promise-keeping constraint. Equation (D.2) implies that the largest possible atom consistent with $q_{\tau-}$ is $(q_{\tau-} - \underline{q})/(1-\underline{q})$, which corresponds to the atom in Theorem 1. On the other hand, once the incentive compatibility constraint is binding, equation (D.1) implies that the largest monitoring rate consistent with the promise-keeping and the incentive compatibility constraint is m^*. Thus, because the benefit of monitoring is increasing over time, the optimal policy requires to perform as much monitoring as possible once it becomes profitable to do so. Hence, the support of the monitoring distribution is either a singleton (deterministic monitoring) or an interval $[\hat{\tau}, \infty]$.

First, notice that any atom has to be of size

$$\Delta M_\tau^d = \log\left(\frac{1-\underline{q}}{1-q_{\tau-}}\right),$$

and that the continuation payoff at the atom date satisfies

$$U_{\tau-} = \left(\frac{1-q_{\tau-}}{1-\underline{q}}\right) U_\tau + \left(\frac{q_{\tau-} - \underline{q}}{1-\underline{q}}\right) \mathcal{M}(\mathbf{U}, x_\tau^\theta)$$

Whenever the IC constraint is binding on an interval of time, the monitoring rate is given by

$$m = (r+\lambda)\frac{\underline{q}}{1-\underline{q}}.$$

The payoff at time zero of a policy with monitoring at a rate m in $[0, \tilde{\tau})$ and an atom at time $\hat{\tau} = \tilde{\tau} + \delta$ is

$$\begin{aligned}\mathcal{U}(\tilde{\tau}, \delta) = &\int_0^{\tilde{\tau}} e^{-(r+m)\tau}\left(u(x_\tau^\theta) + m\mathcal{M}(\mathbf{U}, x_\tau^\theta)\right)\mathrm{d}\tau + \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r\tau - m\hat{\tau}} u(x_\tau^\theta)\mathrm{d}\tau \\ &+ e^{-r(\tilde{\tau}+\delta)-m\tilde{\tau}}\left[\left(\frac{1-q_{\tilde{\tau}+\delta-}}{1-\underline{q}}\right) U_{\tilde{\tau}+\delta} + \left(\frac{q_{\tilde{\tau}+\delta-} - \underline{q}}{1-\underline{q}}\right)\mathcal{M}(\mathbf{U}, x_{\tilde{\tau}+\delta}^\theta)\right] \quad \text{(D.13)}\end{aligned}$$

where

$$U_{\tilde{\tau}+\delta} = \int_{\tilde{\tau}+\delta}^{\infty} e^{-(r+m)(\tau-\tilde{\tau}-\delta)}\left(u(x_\tau^\theta) + m\mathcal{M}(\mathbf{U}, x_\tau^\theta)\right)\mathrm{d}\tau$$

Suppose that the IC constraint is binding at time 0, that is assume that $q_0 = \underline{q}$, then we have that

$$q_{\tilde{\tau}+\delta-} = e^{(r+\lambda)\delta}\underline{q},$$

which means that δ must satisfy

$$\delta \leq \frac{1}{r+\lambda} \log \frac{1}{\underline{q}}.$$

Replacing $q_{\tilde{\tau}+\delta-}$ in (D.13) we get

$$\mathcal{U}(\tilde{\tau}, \delta) = \int_0^{\tilde{\tau}} e^{-(r+m)\tau} \left(u(x_\tau^\theta) + m\mathcal{M}(\mathbf{U}, x_\tau^\theta)\right) \mathrm{d}\tau + \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r\tau - m\tilde{\tau}} u(x_\tau^\theta) \mathrm{d}\tau$$
$$+ e^{-r(\tilde{\tau}+\delta)-m\tilde{\tau}} \left[\left(\frac{1 - e^{(r+\lambda)\delta}\underline{q}}{1-\underline{q}} \right) U_{\tilde{\tau}+\delta} + \left(\frac{e^{(r+\lambda)\delta} - 1}{1-\underline{q}} \right) \underline{q}\mathcal{M}(\mathbf{U}, x_{\tilde{\tau}+\delta}^\theta) \right] \quad \text{(D.14)}$$

Next, we show that for any given δ we have that $\partial \mathcal{U}(\tilde{\tau}, \delta)/\partial\tilde{\tau} = 0 \Rightarrow \partial^2 \mathcal{U}(\tilde{\tau}, \delta)/\partial\tilde{\tau}^2 > 0$. This means that the maximum cannot have an interior value for $\tilde{\tau}$.

Differentiating (D.14) we get

$$\frac{\partial}{\partial\tilde{\tau}} \mathcal{U}(\tilde{\tau}, \delta) = e^{-(r+m)\tilde{\tau}} \left(u(x_{\tilde{\tau}}^\theta) + m\mathcal{M}(\mathbf{U}, x_{\tilde{\tau}}^\theta)\right) + e^{-r(\tilde{\tau}+\delta)-m\tilde{\tau}} u(x_{\tilde{\tau}+\delta}^\theta) - e^{-(r+m)\tilde{\tau}} u(x_{\tilde{\tau}}^\theta)$$
$$-m \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r\tau - m\tilde{\tau}} u(x_\tau^\theta) \mathrm{d}\tau - (r+m) e^{-r(\tilde{\tau}+\delta)-m\tilde{\tau}} \left[\left(\frac{1 - e^{(r+\lambda)\delta}\underline{q}}{1-\underline{q}} \right) U_{\tilde{\tau}+\delta} + \left(\frac{e^{(r+\lambda)\delta} - 1}{1-\underline{q}} \right) \underline{q}\mathcal{M}(\mathbf{U}, x_{\tilde{\tau}+\delta}^\theta) \right]$$
$$+ e^{-r(\tilde{\tau}+\delta)-m\tilde{\tau}} \left[\left(\frac{1 - e^{(r+\lambda)\delta}\underline{q}}{1-\underline{q}} \right) \frac{\partial}{\partial\tilde{\tau}} U_{\tilde{\tau}+\delta} + \left(\frac{e^{(r+\lambda)\delta} - 1}{1-\underline{q}} \right) \underline{q}\dot{x}_{\tilde{\tau}+\delta}^\theta (U_H - U_L) \right]$$

where

$$\frac{\partial}{\partial\tilde{\tau}} U_{\tilde{\tau}+\delta} = -u(x_{\tilde{\tau}+\delta}^\theta) - m\mathcal{M}(\mathbf{U}, x_{\tilde{\tau}+\delta}^\theta) + (r+m) U_{\tilde{\tau}+\delta}$$

Rearranging terms we get

$$
\begin{aligned}
\frac{\partial}{\partial \tilde{\tau}}\mathcal{U}(\tilde{\tau},\delta) &= e^{-(r+m)\tilde{\tau}}\Big[m\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}}) + e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) \\
&- m\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - (r+m)\left(\frac{e^{\lambda\delta}-e^{-r\delta}}{1-\underline{q}}\right)\underline{q}\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) \\
&- \left(\frac{e^{-r\delta}-e^{\lambda\delta}\underline{q}}{1-\underline{q}}\right)\Big(u(x^{\theta}_{\tilde{\tau}+\delta}) + m\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta})\Big) + \left(\frac{e^{\lambda\delta}-e^{-r\delta}}{1-\underline{q}}\right)\underline{q}\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H-U_L)\Big] \\
&= e^{-(r+m)\tilde{\tau}}\Big[m\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}}) + (e^{\lambda\delta}-e^{-r\delta})\left(\frac{\underline{q}}{1-\underline{q}}\right)u(x^{\theta}_{\tilde{\tau}+\delta}) \\
&- m\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - (r+m)(e^{\lambda\delta}-e^{-r\delta})\left(\frac{\underline{q}}{1-\underline{q}}\right)\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) \\
&- \left(\frac{e^{-r\delta}-e^{\lambda\delta}\underline{q}}{1-\underline{q}}\right)m\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) + (e^{\lambda\delta}-e^{-r\delta})\left(\frac{\underline{q}}{1-\underline{q}}\right)\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H-U_L)\Big] \\
&= e^{-(r+m)\tilde{\tau}}m\Big[\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}}) + \frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}u(x^{\theta}_{\tilde{\tau}+\delta}) \\
&- \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - (r+m)\frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) \\
&- \left(\frac{e^{-r\delta}-e^{\lambda\delta}\underline{q}}{1-\underline{q}}\right)\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) + \frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H-U_L)\Big]
\end{aligned}
$$

So, finally, we can write

$$
\begin{aligned}
&\frac{\partial}{\partial \tilde{\tau}}\mathcal{U}(\tilde{\tau},\delta) = e^{-(r+m)\tilde{\tau}}m\Big[\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}}) + \frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}u(x^{\theta}_{\tilde{\tau}+\delta}) \\
&- \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - \left(\frac{r}{r+\lambda}e^{\lambda\delta} + \frac{\lambda}{r+\lambda}e^{-r\delta}\right)\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) + \frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H-U_L)\Big]
\end{aligned}
$$

Let's define

$$
\begin{aligned}
&G(\tilde{\tau}) = \mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}}) + \frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}u(x^{\theta}_{\tilde{\tau}+\delta}) \\
&- \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - \left(\frac{r}{r+\lambda}e^{\lambda\delta} + \frac{\lambda}{r+\lambda}e^{-r\delta}\right)\mathcal{M}(\mathbf{U},x^{\theta}_{\tilde{\tau}+\delta}) + \frac{(e^{\lambda\delta}-e^{-r\delta})}{r+\lambda}\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H-U_L)
\end{aligned}
$$

So

$$
\frac{\partial}{\partial \tilde{\tau}}\mathcal{U}(\tilde{\tau},\delta) = e^{-(r+m)\tilde{\tau}}mG(\tilde{\tau})
$$

Clearly, the first order condition is satisfied only if $G(\tilde{\tau}) = 0$. Moreover, $G(\tilde{\tau}) = 0$ implies that

$\frac{\partial^2}{\partial \tilde{\tau}^2}\mathcal{U}(\tilde{\tau},\delta) = G'(\tilde{\tau})$. Differentiating $G(\tilde{\tau})$ we get

$$\begin{aligned}
G'(\tilde{\tau}) &= \dot{x}^{\theta}_{\tilde{\tau}}(U_H - U_L) + \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda} u'(x^{\theta}_{\tilde{\tau}+\delta})\dot{x}^{\theta}_{\tilde{\tau}+\delta} - e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) + u(x^{\theta}_{\tilde{\tau}}) \\
&- r\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - \left(\frac{r}{r+\lambda}e^{\lambda\delta} + \frac{\lambda}{r+\lambda}e^{-r\delta}\right)\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H - U_L) + \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda}\ddot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H - U_L) \\
&= \dot{x}^{\theta}_{\tilde{\tau}}(U_H - U_L) + \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda} u'(x^{\theta}_{\tilde{\tau}+\delta})\dot{x}^{\theta}_{\tilde{\tau}+\delta} - e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) + u(x^{\theta}_{\tilde{\tau}}) \\
&- r\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau - \left(\frac{r}{r+\lambda}e^{\lambda\delta} + \frac{\lambda}{r+\lambda}e^{-r\delta}\right)\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H - U_L) - \lambda\frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda}\dot{x}^{\theta}_{\tilde{\tau}+\delta}(U_H - U_L) \\
&= \left(\dot{x}^{\theta}_{\tilde{\tau}} - e^{\lambda\delta}\dot{x}^{\theta}_{\tilde{\tau}+\delta}\right)(U_H - U_L) + \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda} u'(x^{\theta}_{\tilde{\tau}+\delta})\dot{x}^{\theta}_{\tilde{\tau}+\delta} - e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) + u(x^{\theta}_{\tilde{\tau}}) \\
&- r\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau
\end{aligned}$$

Noting that

$$\frac{\partial}{\partial\delta}e^{\lambda\delta}\dot{x}^{\theta}_{\tilde{\tau}+\delta} = \lambda e^{\lambda\delta}\dot{x}^{\theta}_{\tilde{\tau}+\delta} + e^{\lambda\delta}\ddot{x}^{\theta}_{\tilde{\tau}+\delta} = \lambda e^{\lambda\delta}\dot{x}^{\theta}_{\tilde{\tau}+\delta} - \lambda e^{\lambda\delta}\dot{x}^{\theta}_{\tilde{\tau}+\delta} = 0$$

we conclude that

$$G'(\tilde{\tau}) = \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda} u'(x^{\theta}_{\tilde{\tau}+\delta})\dot{x}^{\theta}_{\tilde{\tau}+\delta} - e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) + u(x^{\theta}_{\tilde{\tau}}) - r\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau \qquad \text{(D.15)}$$

Using integration by parts we find that

$$-r\int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u(x^{\theta}_{\tau})\mathrm{d}\tau = e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) - u(x^{\theta}_{\tilde{\tau}}) - \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u'(x^{\theta}_{\tau})\dot{x}^{\theta}_{\tau}\mathrm{d}\tau$$

Replacing in equation (D.15) we arrive to

$$\begin{aligned}
G'(\tilde{\tau}) &= \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda} u'(x^{\theta}_{\tilde{\tau}+\delta})\dot{x}^{\theta}_{\tilde{\tau}+\delta} - e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) + u(x^{\theta}_{\tilde{\tau}}) + e^{-r\delta}u(x^{\theta}_{\tilde{\tau}+\delta}) - u(x^{\theta}_{\tilde{\tau}}) - \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u'(x^{\theta}_{\tau})\dot{x}^{\theta}_{\tau}\mathrm{d}\tau \\
&= \frac{(e^{\lambda\delta} - e^{-r\delta})}{r+\lambda} u'(x^{\theta}_{\tilde{\tau}+\delta})\dot{x}^{\theta}_{\tilde{\tau}+\delta} - \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-r(\tau-\tilde{\tau})}u'(x^{\theta}_{\tau})\dot{x}^{\theta}_{\tau}\mathrm{d}\tau \qquad \text{(D.16)}
\end{aligned}$$

Replacing $\dot{x}^{\theta}_{\tau} = \lambda(\bar{a}-\theta)e^{-\lambda\tau}$ in equation (D.16) we get

$$G'(\tilde{\tau}) = \lambda(\bar{a}-\theta)e^{-\lambda\tilde{\tau}}\left[\frac{(1 - e^{-(r+\lambda)\delta})}{r+\lambda}u'(x^{\theta}_{\tilde{\tau}+\delta}) - \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})}u'(x^{\theta}_{\tau})\mathrm{d}\tau\right] \qquad \text{(D.17)}$$

On the one hand, if $\theta = 0$, then we have that $u'(x^\theta_{\tilde{\tau}+\delta}) > u'(x^\theta_\tau)$ for all $\tilde{\tau} + \delta > \tau$, which means that

$$\begin{aligned}
G'(\tilde{\tau}) &= \lambda(\bar{a} - \theta)e^{-\lambda\tilde{\tau}} \left[\frac{(1 - e^{-(r+\lambda)\delta})}{r + \lambda} u'(x^\theta_{\tilde{\tau}+\delta}) - \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})} u'(x^\theta_\tau) \mathrm{d}\tau \right] \\
&> \lambda(\bar{a} - \theta)e^{-\lambda\tilde{\tau}} \left[\frac{(1 - e^{-(r+\lambda)\delta})}{r + \lambda} u'(x^\theta_{\tilde{\tau}+\delta}) - u'(x^\theta_{\tilde{\tau}+\delta}) \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})} \mathrm{d}\tau \right] \\
&= \lambda(\bar{a} - \theta)e^{-\lambda\tilde{\tau}} \left[\frac{(1 - e^{-(r+\lambda)\delta})}{r + \lambda} u'(x^\theta_{\tilde{\tau}+\delta}) - u'(x^\theta_{\tilde{\tau}+\delta}) \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})} \mathrm{d}\tau \right] \\
&= 0.
\end{aligned}$$

On the other hand, if $\theta = 1$, then we have that $u'(x^\theta_{\tilde{\tau}+\delta}) < u'(x^\theta_\tau)$ for all $\tilde{\tau} + \delta > \tau$, which means that

$$\begin{aligned}
G'(\tilde{\tau}) &= \lambda(\bar{a} - \theta)e^{-\lambda\tilde{\tau}} \left[\frac{(1 - e^{-(r+\lambda)\delta})}{r + \lambda} u'(x^\theta_{\tilde{\tau}+\delta}) - \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})} u'(x^\theta_\tau) \mathrm{d}\tau \right] \\
&= \lambda(\theta - \bar{a})e^{-\lambda\tilde{\tau}} \left[-\frac{(1 - e^{-(r+\lambda)\delta})}{r + \lambda} u'(x^\theta_{\tilde{\tau}+\delta}) + \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})} u'(x^\theta_\tau) \mathrm{d}\tau \right] \\
&> \lambda(\theta - \bar{a})e^{-\lambda\tilde{\tau}} \left[-\frac{(1 - e^{-(r+\lambda)\delta})}{r + \lambda} u'(x^\theta_{\tilde{\tau}+\delta}) + u'(x^\theta_{\tilde{\tau}+\delta}) \int_{\tilde{\tau}}^{\tilde{\tau}+\delta} e^{-(r+\lambda)(\tau-\tilde{\tau})} di\tau \right] \\
&= 0.
\end{aligned}$$

This means that, for any $\delta \geq 0$, we have $\frac{\partial}{\partial\tilde{\tau}}\mathcal{U}(\tilde{\tau}, \delta) = 0$ implies $\frac{\partial^2}{\partial\tilde{\tau}^2}\mathcal{U}(\tilde{\tau}, \delta) > 0$ which means that the optimal monitoring policy can not have an interior $\tilde{\tau}$, that is $\tilde{\tau}^* \in \{0, \infty\}$.

$\square$

E Model with Exogenous News

In this appendix, we consider the model with exogenous news. Thus far, we have ignored alternative sources of information, besides monitoring. In this section, we explore the effect of having exogenous news on the optimal monitoring policy. We show that exogenous news, not only crowd-out monitoring but by altering the severity of the moral hazard issue across states, may modify the monitoring policy in a significant way.

Exogenous news such as media articles, customer reviews, and academic research provide information to the market that may complement or substitute the principal's own monitoring efforts. To provide some insights about the interaction between monitoring and news, we consider the presence of an exogenous news process that may reveal current quality to the market. More specifically, we consider the case in which the quality of the product is revealed to the market at a Poisson arrival rate.

Assume there are two Poisson processes $(N_t^L)_{t\geq 0}$ and $(N_t^H)_{t\geq 0}$. The process N_t^L is a bad news process with mean arrival rate $\theta_t = \mu_L \mathbf{1}_{\{\theta_t = L\}}$, and N_t^H is a good news process with mean arrival rate $\mu_H \mathbf{1}_{\{\theta_t = H\}}$. When $\mu_L \neq \mu_H$ we say that news are *asymmetric*, in which case, the absence of news is informative about the firm quality. On the other hand, if $\mu_L = \mu_H$ the lack of news arrival is uninformative. We say that we are in the *bad news* case when $\mu_L > \mu_H$ and in the *good news* case if $\mu_H > \mu_L$. In the absence of exogenous news and monitoring, beliefs evolve according to

$$\dot{x}_t = \lambda(a_t - x_t) - (\mu_H - \mu_L)x_t(1 - x_t).$$

The second term cancels if $\mu_H = \mu_L$ and the dynamics of beliefs (in the absence of any monitoring by the principal and arrival of exogenous news) is the same as in the case without news. On the other hand, if $\mu_L \neq \mu_H$, the exogenous news introduces a new term in the drift of reputation. That term is positive in the bad news case and negative in the good news case. The market learns from the absence of news since no news is informative when the news processes have asymmetric arrival rates.

Let's first consider the case with symmetric news arrival, i.e. $\mu_L = \mu_H = \mu$. From the firm's point of view, it does not matter if the state is learned due to monitoring or exogenous news. The only difference is that now, there is an extra arrival rate that reveals the state. If we denote the date at which quality is revealed, either by monitoring or exogenous news, by $\tilde{T}_n$, then we can still write the incentive compatibility constraint as

$$E\left[e^{-(r+\lambda)(\tilde{T}_n - t)}\middle|\mathcal{F}_t\right] \geq \underline{q}.$$

This means that we can still use q_τ as our main state variable, and the dynamics of q_τ are given by

$$\mathrm{d}q_\tau = (r+\lambda)q_\tau \mathrm{d}\tau - (1 - q_\tau)(\mathrm{d}M_\tau^c + \mu \mathrm{d}t). \tag{E.1}$$

Notice that the only difference between equations (D.1) and (E.1) is that the monitoring rate $\mathrm{d}M_\tau^c$

is incremented by $\mu \mathrm{d}\tau$ due to the exogenous news. Similarly, because the problem of the principal is the same going forward no matter if quality was learned due to monitoring or exogenous news, we can still write the problem recursively based on the time elapsed since the last time the firm type was observed (either by monitoring or news) and the type observed at that time. The principal's continuation value now evolves according to

$$\mathrm{d}U_\tau = \left(rU_\tau - u(x_\tau^\theta)\right)\mathrm{d}\tau + \left(U_\tau - \mathcal{M}(\mathbf{U}, x_\tau^\theta)\right)\mathrm{d}M_\tau^c + \mu\left(U_\tau - x_\tau^\theta U_H - (1 - x_\tau^\theta)U_L\right)\mathrm{d}\tau. \quad \text{(E.2)}$$

Hence, the Principal's problem has the same structure as before, with the exception that now the principal gets some monitoring with intensity μ *for free.* When news arrivals are symmetric, exogenous news is a perfect substitute for monitoring. Lemmas 8 and 4.4 still apply, and the monitoring rate is positive only if the incentive compatibility constraint is binding, in which case $\mathrm{d}q_\tau = 0$ so the monitoring rate is

$$m^* + \mu = (r + \lambda)\frac{\underline{q}}{1 - \underline{q}}.$$

Clearly, the monitoring rate to keep the incentive constraint binding needs to be positive only if μ is low enough. Otherwise, exogenous news suffices for incentive purposes. In this latter case, exogenous news are enough to discipline the firm, and the only purpose of monitoring is to learn the sate. Depending on the magnitude of μ, the optimal monitoring policy may entail some or no random monitoring. We have the following proposition, which is a direct implication of Proposition 1.

Proposition E.1. *Suppose that $\mu_L = \mu_H$. If $(r + \lambda)\frac{\underline{q}}{1-\underline{q}} \geq \mu$ then the optimal monitoring policy is the one characterized in Propositions 3 and 1 with a Poisson monitoring rate.*

$$m^* = (r + \lambda)\frac{\underline{q}}{1 - \underline{q}} - \mu.$$

On the other hand, if $(r + \lambda)\frac{\underline{q}}{1-\underline{q}} < \mu$, then the optimal monitoring policy is deterministic.

Proof. Letting $\tilde{M}_\tau^c = M_\tau^c + \mu\tau$ and $\tilde{u}(x) = u(x) + \mu c$, we can write

$$\begin{aligned}\mathrm{d}q_\tau &= (r + \lambda)q_\tau \mathrm{d}\tau - (1 - q_\tau)\mathrm{d}\tilde{M}_\tau^c \\ \mathrm{d}U_\tau &= \left(rU_\tau - \tilde{u}(x_\tau^\theta)\right)\mathrm{d}\tau + \left(U_\tau - \mathcal{M}(\mathbf{U}, x_\tau^\theta)\right)\mathrm{d}\tilde{M}_\tau^c,\end{aligned}$$

so the optimal control problem follows the same structure as before with two diferences: (1) now $\mathrm{d}\tilde{M}_\tau^c$ must be greater or equal than $\mu \mathrm{d}\tau$, and (2) q_τ is bounded below by $\frac{\mu}{r+\lambda+\mu}$. If $(r + \lambda)\frac{\underline{q}}{1-\underline{q}} \geq \mu$ then (1) and (2) are not binding. On the other hand, if $(r + \lambda)\frac{\underline{q}}{1-\underline{q}} < \mu$ then $q_\tau > \underline{q}$. Hence, the incentive compatibility constraint is slack at all times, so the solution to the Principal problem corresponds to the one in Section A, which means that monitoring is deterministic. ☐

E.1 Asymmetric News Intensity

The qualitative results are different if $\mu_H \neq \mu_L$. In this case, the presence of news changes the dynamics of incentives: the monitoring rate changes over time and is dependent on the outcome of the outcome in the last review. We do not solve the full problem here and instead focus on the case in which the principal's preferences are linear. Based on our previous analysis of the linear, it is natural to conjecture that the optimal policy has (1) no atoms and that (2) the monitoring rate is positive only if the incentive compatibility constraint is binding. We can use the maximum principle to verify if our conjectured policy is optimal. We relegate a detailed discussion of the solution to the appendix.

We focus on the simplest case with parameters such that the optimal policy has $m_\tau > 0$ for all $\tau \geq 0$; this case illustrates the effect of introducing exogenous news on the optimal monitoring policy at the lowest cost of technical complications.[3]

E.2 Incentive Compatibility and the Principal's Problem with News

In the presence of exogenous news, we cannot use a single state variable to characterize incentive compatibility. With persistent state variables, we need additional state variables to keep track of the continuation value across states. As in Fernandes and Phelan (2000) we use the continuation value conditional on the firm's private information (i.e., the firm quality).

Let Π_τ^θ be the firm's continuation value conditional on being type θ_τ and define $D_\tau \equiv \Pi_\tau^H - \Pi_\tau^L$. The continuation value must satisfy the Bellman equations

$$
\begin{aligned}
r\Pi_\tau^H &= \max_{a\in[0,\bar{a}]} \left\{ x_\tau - ka_\tau - \lambda(1-a_\tau)D_{\tau^-} + (\mu_H + m_\tau)(\Pi(H) - \Pi_\tau^H) + \dot{\Pi}_\tau^H \right\} \\
r\Pi_\tau^L &= \max_{a\in[0,\bar{a}]} \left\{ x_\tau - ka_\tau + \lambda a_\tau D_{\tau^-} + (\mu_L + m_\tau)(\Pi(L) - \Pi_\tau^L) + \dot{\Pi}_\tau^L \right\},
\end{aligned}
$$

where we use the fact that if $a_t = \bar{a}$ for any $t \geq T_n$ then, given $\theta_{T_n} = \theta$, the continuation payoff is $\Pi_0^\theta = \Pi(\theta)$ (recall that $\Pi(\theta)$ is given by (1)). From here it follows that full effort $a_\tau = \bar{a}$ is incentive compatible if and only if:[4]

$$
D_\tau \geq \frac{k}{\lambda}.
$$

The evolution of D_τ can be derived (analogously to what we have done before) to be

$$
\dot{D}_\tau = (r + \lambda + m_\tau)D_\tau - \mu_H(\Pi(H) - \Pi_\tau^H) + \mu_L(\Pi(L) - \Pi_\tau^L) - m_\tau\Delta.
$$

with a boundary condition $D_{\bar{\tau}} = \Delta \equiv \Pi(H) - \Pi(L) = 1/(r+\lambda)$.

[3] Such policy is optimal when the rates of exogenous news arrivals are low. When those rates are large, after some histories the principal will not monitor at all since the exogenous news would be sufficient to provide incentives, as in Board and Meyer-ter-Vehn (2013). That is, our analysis focuses on the cases where news are not informative enough, and so some amount of monitoring is needed at all times to solve the agency problem.

[4] This incentive compatibility is analogous to that in Board and Meyer-ter-Vehn (2013) except that there the only source of information is the exogenous news process and we allow for additional information from costly inspections.

From the principal's viewpoint, it does not matter whether he learns the state due to monitoring or exogenous news. In either case, the problem facing the principal is the same going forward. Hence, we can write the problem recursively using as state variables both the time elapsed since the last time the firm type was observed (either by monitoring or news), and the type observed at that time. The optimal control problem (ignoring jumps in the monitoring distribution) becomes

$$
\begin{aligned}
\mathscr{G}^\theta(\mathbf{U}) = \sup_{\bar{\tau}, m_\tau, \Pi_0^{\neg\theta}} & \int_0^{\bar{\tau}} e^{-r\tau - M_{\tau^-}} \Big(x_\tau^\theta + \mu_H x_\tau^\theta U_H + \mu_L (1 - x_\tau^\theta) U_L + m_\tau \mathcal{M}(\mathbf{U}, x_\tau) \Big) \, \mathrm{d}\tau \\
& + e^{-r\bar{\tau} - M_{\bar{\tau}}} \mathcal{M}(\mathbf{U}, x_{\bar{\tau}}) \\
& \text{subject to} \\
& \dot{\Pi}_\tau^H = (r + \mu_H + m_\tau)\Pi_\tau^H - x_\tau + k\bar{a} + \lambda(1 - \bar{a})(\Pi_\tau^H - \Pi_\tau^L) - (\mu_H + m_\tau)\Pi(H), \; \Pi_{\bar{\tau}}^H = \Pi(H) \\
& \dot{\Pi}_\tau^L = (r + \mu_L + m_\tau)\Pi_\tau^L - x_\tau + k\bar{a} - \lambda\bar{a}(\Pi_\tau^H - \Pi_\tau^L) - (\mu_L + m_\tau)\Pi(L), \; \Pi_{\bar{\tau}}^L = \Pi(L) \\
& \Pi_0^\theta = \Pi(\theta) \\
& \frac{k}{\lambda} \le \Pi_\tau^H - \Pi_\tau^L, \; \forall \tau \in [0, \bar{\tau}] \\
& 0 \le m_\tau.
\end{aligned}
$$

Note that in the previous formulation, the continuation payoff given the counterfactual type $\neg\theta$ (if $\theta = H$ then $\neg\theta = L$ and vice versa), which we denote by $\Pi_0^{\neg\theta}$, is not given by $\Pi(\neg\theta)$. The solution to this problem critically depends on the intensity of bad versus good news arrivals. We first consider the symmetric case.

We consider the asymmetric case, $\mu_H \neq \mu_L$, so that the intensity of news arrival depends on the firm's quality. Such asymmetry seems natural: in some industries and under some market conditions, good news tend to be revealed faster than bad news, among other things, because firms themselves may delay the release of bad news. Sometimes, bad news tend to be revealed faster than good news, perhaps because news agencies and TV broadcasts face stronger demand for bad news stories.

The main question we address here is how monitoring rates are affected by reputation when exogenous news are asymmetric. We do not solve the full problem here, and instead we focus on the case in which the principal's preferences are linear. Based on our previous analysis, it is natural to conjecture that the optimal policy has 1) no atoms in the distribution of monitoring (in particular, $\bar{\tau} = \infty$), and 2) the monitoring rate is positive (i.e., $m_\tau > 0$) only if the incentive compatibility constraint is binding, that is if $\Pi_\tau^H - \Pi_\tau^L = k/\lambda$. We can use the maximum principle to verify if our conjectured policy is optimal. We relegate a detailed discussion of the optimality conditions to the appendix.

Given this monitoring policy, we can follow the same steps as before, and derive the monitoring rate using the incentive compatibility constraint: $(\dot{\Pi}_\tau^H - \dot{\Pi}_\tau^L) = 0$ and $\Pi_\tau^H - \Pi_\tau^L = k/\lambda$. These conditions are necessary for the incentive compatibility constraints to bind at all times. They imply:

$$m_\tau = \alpha + \beta \Pi_\tau^L, \tag{E.3}$$

where

$$\alpha = \frac{(r+\lambda)k/\lambda + \mu_H(k/\lambda - \Pi(H)) + \mu_L \Pi(L)}{\Delta - k/\lambda}$$
$$\beta = \frac{\mu_H - \mu_L}{\Delta - k/\lambda}.$$

The constant β is positive in the good news case and negative otherwise so in the bad news case the monitoring rate is positive only if $\Pi_\tau^L \leq -\alpha/\beta$, and in the good news case, the monitoring rate is positive only if $\Pi_\tau^L \geq -\alpha/\beta$. That is, with bad news, monitoring is needed only if the firm's continuation value is low, and with good news, monitoring is needed only if the firm's continuation value is high. The logic for these conditions follows the results in Board and Meyer-ter-Vehn (2013): With bad news, the incentives for effort increase in reputation, while with good news the incentives for effort decrease in reputation.

We focus on the simplest case with parameters such that the optimal policy has $m_\tau > 0$ for all $\tau \geq 0$; this case illustrates the effect of introducing exogenous news on the optimal monitoring policy at the lowest cost of technical complications.[5] Using the relation $\Pi_\tau^H = \Pi_\tau^L + D_\tau = \Pi_\tau^L + k/\lambda$ and the monitoring rate (E.3) we write the evolution of the low quality firm continuation value as

$$\dot{\Pi}_\tau^L = -(\mu_L + \alpha)\Pi(L) + (r + \mu_L + \alpha - \beta\Pi(L))\Pi_\tau^L + \beta(\Pi_\tau^L)^2 - x_\tau. \tag{E.4}$$

If $\theta_0 = L$ then the initial condition is $\Pi_0^L = \Pi(L)$. If $\theta_0 = H$ (and the incentive compatibility is binding) the initial condition is $\Pi_0^L = \Pi(H) - k/\lambda$.[6] We can analyze the evolution of monitoring by studying the phase diagram in the space (x_τ, Π_τ^L) in Figure 2.

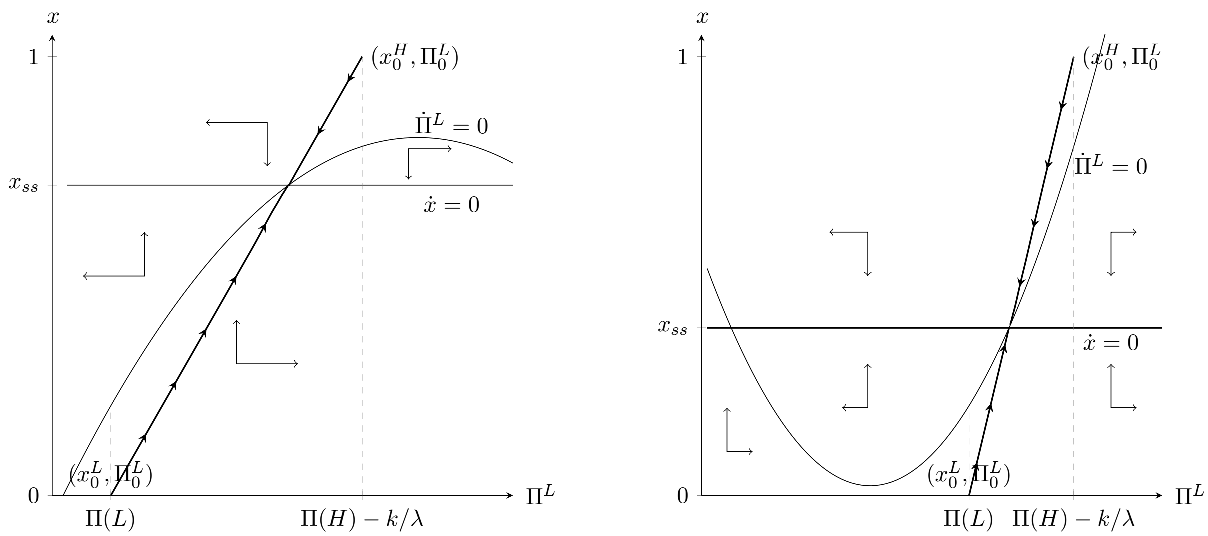
Using the ODE for Π_τ^L in equation (E.4) we get a quadratic equation for the steady state:

$$0 = -(\mu_L + \alpha)\Pi(L) + (r + \mu_L + \alpha - \beta\Pi(L))\Pi^L + \beta(\Pi^L)^2 - x. \tag{E.5}$$

This quadratic equation has two solutions. We show that in the good news case only the largest solution is consistent with a positive monitoring rate, while in the bad news only the smallest one is consistent with a positive monitoring rate. So if the solution has positive monitoring rate at all times, then the solution must correspond to the saddle point trajectory in the phase diagram in Figure 2.

[5] Such policy is optimal when the rates of exogenous news arrivals are low. When those rates are large, after some histories the principal will not monitor at all since the exogenous news would be sufficient to provide incentives, as in Board and Meyer-ter-Vehn (2013). That is, our analysis focuses on the cases where news are not informative enough, and so some amount of monitoring is needed at all times to solve the agency problem.

[6] If the IC constraint is not binding at time zero then the initial value must be computed indirectly.



(a) Case with bad news

(b) Case with good news

Figure 2: Phase diagram. The (x_τ, Π_τ^L) system has two steady states. In each case, one of the steady states is a saddle point. If the optimal solution is such that $m_\tau > 0$ all $\tau \geq 0$, then the optimal solution corresponds to the trajectory converging to the saddle point. In this case, the analysis of the phase diagram reveals that the trajectory of Π_τ^L must be monotone between news arrivals. This immediately implies that the evolution of monitoring between news is monotone as well.

From inspection of the phase diagram, it is clear that Π_τ^L is monotone: it starts decreasing after good news and starts increasing after bad news. This implies the dynamics of optimal monitoring that are described in Figure 3. In the bad news case, monitoring increases after (bad) news. The opposite is optimal in the good news case. As previously mentioned, this is driven by the dynamics of reputational incentives. In the bad news case, incentives weaken as reputation goes down. As Board and Meyer-ter-Vehn (2013) point out, a high reputation firm has more to lose from a collapse in its reputation following a breakdown than a low reputation firm. Hence, inspections are most needed for incentive purposes when reputation is low. In the good news case, incentives decrease in reputation; a low reputation firm has more to gain from a breakthrough that boosts its reputation than a high reputation firm. In the good news case, inspections are thus most needed when reputation is high. Accordingly, monitoring complements exogenous news, being used when exogenous news are ineffective at providing incentives. We still need to verify that: (1) the optimal monitoring policy is optimal, and (2) show that the dynamics of the firm's continuation value satisfy the monotonicity properties in Figure 2. We consider the optimality conditions for the optimal policy in Section E.3 and study the steady states of the firm's continuation payoffs in Section E.4.

E.3 Necessary Conditions with Asymmetric News

The Hamiltonian for the optimal control problem is

$$\begin{aligned}\mathcal{H}(\Pi_\tau^L,\Pi_\tau^H,\zeta_\tau,\nu_\tau^L,\nu_\tau^H,\psi_\tau,m_\tau,\tau) = \zeta_\tau((r+m_\tau)U_\tau^\theta - x_\tau^\theta - \mu_H x_t^\theta U_H - \mu_L(1-x_\tau^\theta)U_L - m_\tau \mathcal{M}(\mathbf{U},x_\tau)\\ + \psi_\tau(\Pi_\tau^H - \Pi_\tau^L - k/\lambda) + \nu_\tau^H\big((r+\mu_H+m_\tau)\Pi_\tau^H - x_\tau + k\bar{a} + \lambda(1-\bar{a})(\Pi_\tau^H - \Pi_\tau^L)\\ - (\mu_H+m_\tau)\Pi(H)\big) + \nu_\tau^L\big((r+\mu_L+m_\tau)\Pi_\tau^L - x_\tau + k\bar{a} - \lambda\bar{a}(\Pi_\tau^H - \Pi_\tau^L)\\ - (\mu_L+m_\tau)\Pi(L)\big)\end{aligned}$$

As before, we have that $\zeta_\tau - = 1$ and the evolution of the remaining co-state variables is The evolution of the co-state variables is given by

$$\begin{aligned}\dot{\nu}_\tau^H &= -(\mu_H + \lambda(1-\bar{a}))\nu_\tau^H - \psi_\tau + \lambda\bar{a}\nu_\tau^L\\ \dot{\nu}_\tau^L &= -(\mu_L + \lambda\bar{a})\nu_\tau^L + \psi_\tau + \lambda(1-\bar{a})\nu_\tau^H.\end{aligned}$$

The switching function $S(\tau)$ is given by

$$S(\tau) = \mathcal{M}(\mathbf{U},x_\tau) + \nu_\tau^H(\Pi_\tau^H - \Pi(H)) + \nu_\tau^L(\Pi_\tau^L - \Pi(L)) - U_\tau^\theta.$$

We pin-down the boundary condition for the co-state variables ν_τ^θ by looking at the switching function. The rate of monitoring is positive (and finite) at time zero only if $S(0)=0$ which implies that

$$0 = \mathcal{M}(\mathbf{U},\theta) - U_\theta + \nu_0^H(\Pi_0^H - \Pi(H)) + \nu_0^L(\Pi_0^L - \Pi(L)).$$

If the incentive compatibility constraint is binding at time zero, so $\Pi_0^H - \Pi_0^L = k/\lambda$, then when $\theta_0 = L$ and $m_0 > 0$ the initial value of the co-state variable ν_0^H is

$$c = -\nu_0^H\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right).$$

The initial value of the co-state variable ν_0^L is determined by the transversality condition $\lim_{\tau\to\infty}\nu_\tau^L = \nu_{ss}^L$. If the incentive compatibility constraint at time zero were slack (that is $m_0 = 0$) then the initial value would be $\nu_0^H = 0$. The determination of ν_0^L is more complicated in this latter case as ν_τ^L can jump at the junction time τ^m in which the IC constraint becomes binding. Similarly, if $\theta = H$ then we have that ν_0^L is given by

$$c = \nu_0^L\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right)$$

while ν_0^H is determined by the transversality condition $\lim_{\tau\to\infty}\nu_\tau^H = \nu_{ss}^H$. As for $\theta_0 = L$, the same qualification for the case in which the IC constraint is slack at time zero applies. In the same way as we did in the case without news, we can use the condition that the switching function is constant on a singular arc, $\dot{S}_\tau = 0$, to back out the value of the Lagrange multiplier ψ_τ

$$\psi_\tau\big((\Pi^H_\tau - \Pi^L_\tau) - (\Pi(H) - \Pi(L))\big) = \dot{x}^\theta_\tau(U_H - U_L) - \dot{U}^\theta_\tau + (-(\mu_H + \lambda(1-\bar{a}))\nu^H_\tau + \lambda\bar{a}\nu^L_\tau)(\Pi^H_\tau - \Pi(H)) + \nu^H_\tau\dot{\Pi}^H_\tau$$
$$+ (-(\mu_L + \lambda\bar{a})\nu^L_\tau + \lambda(1-\bar{a})\nu^H_\tau)(\Pi^L_\tau - \Pi(L)) + \nu^L_\tau\dot{\Pi}^L_\tau$$

If the incentive compatibility constraint is binding, $\Pi^H_\tau - \Pi^L_\tau = k/\lambda$, then we can write the Lagrange multiplier as

$$\psi_\tau = \frac{1}{k/\lambda - \Delta}\Big[\dot{x}^\theta_\tau(U_H - U_L) - \dot{U}_\tau - \big(\mu_H\nu^H_\tau + \mu_L\nu^L_\tau\big)(\Pi^L_\tau - \Pi(L))$$
$$+ + \big((\mu_H + \lambda(1-\bar{a}))\nu^H_\tau - \lambda\bar{a}\nu^L_\tau\big)\left(\frac{1}{r+\lambda} - \frac{k}{\lambda}\right)(\nu^L_\tau + \nu^H_\tau)\dot{\Pi}^L_\tau\Big].$$

A necessary condition for our conjectured monitoring policy m_τ to be optimal is that the Lagrange multiplier ψ_τ is non-negative whenever the incentive compatibility constraint is binding. The monitoring policy m_τ is positive if and only if this constraint is binding; hence, the condition reduces to verify that $\psi_\tau m_\tau \geq 0$. Given the higher dimensionality of the state space, we can no longer check this condition analytically. However, this condition can be easily verified numerically after solving for the system of ODEs. The Hamiltonian in our problem is not concave, so traditional theorems on the sufficiency of the maximum principle do not apply. However, our problem is a special case of the generalized linear control processes considered by Lansdowne (1970), for which he proves the sufficiency of the maximum principle. The results in Lansdowne (1970) do not apply directly to our problem due to the presence of a state constraint; however, because the state constraint in our problem is linear, his sufficiency result can be extended to our setting.

The dynamics of optimal monitoring are described in Figure 3. In the bad news case, monitoring increases after (bad) news. The opposite is optimal in the good news case. The dynamics of monitoring are driven by the dynamics of reputational incentives. In the bad news case, incentives weaken as reputation goes down. As Board and Meyer-ter-Vehn (2013) point out, a high reputation firm has more to lose from a collapse in its reputation following a breakdown than a low reputation firm. Hence, inspections are most needed for incentive purposes when reputation is low. In the good news case, incentives decrease in reputation; a low reputation firm has more to gain from a breakthrough that boosts its reputation than a high reputation firm. In the good news case, inspections are thus most needed when reputation is high. Accordingly, monitoring complements exogenous news, being used when exogenous news are ineffective at providing incentives.

E.4 Monotonicity of Monitoring Policy with Asymmetric News

Proof. Looking at the phase diagram in Figure 2, we see that if the optimal solution is given by the saddle path, then the trajectory towards the steady state is monotonic, which implies that m_τ is decreasing in x_τ. Hence, we only need to rule out that in the optimal policy the continuation

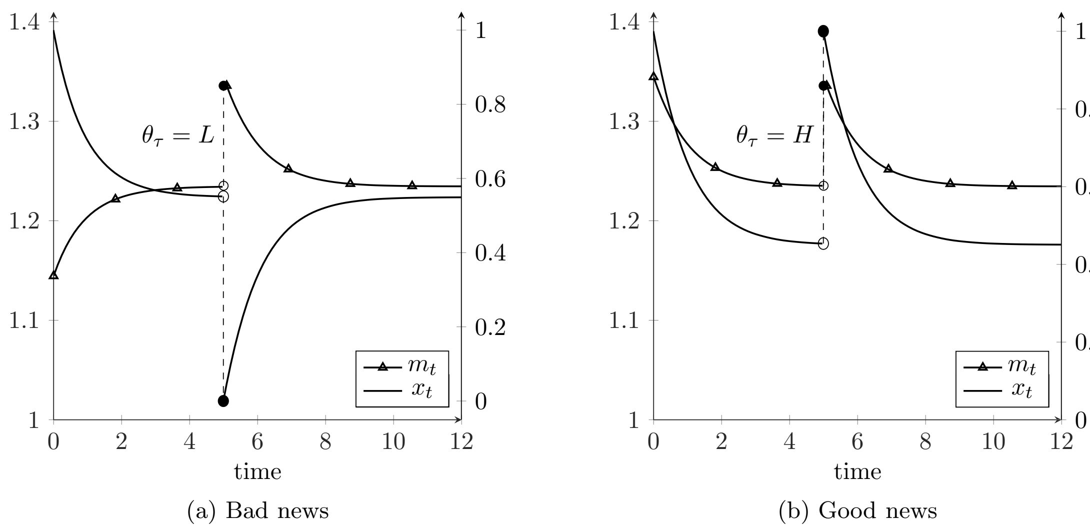


Figure 3: Response of monitoring rates to exogenous news in the bad news and good new cases. In both pictures the starting belief is $x_0 = 1$. The blue curves represent optimal monitoring intensity, m_τ and the red curves the evolution of reputation, x_τ. In the bad news case (left panel) the rate of monitoring increases after negative news (either from inspection or exogenous news). Moreover, optimal monitoring intensity is decreasing in beliefs. The dynamics of monitoring are the opposite in the good news case. Parameters: $r = 0.1$, $k = 0.5$, $c = 0.1$, $\bar{a} = 0.5$, $\lambda = 1$. In the bad news case we take $\mu_H = 0$, and $\mu_L = 0.2$, and in the good news case we take $\mu_H = 0.2$, and $\mu_L = 0$

values converge to the stable steady state. We show this by verifying that the trajectory to the stable steady state violates the non-negativity condition of the monitoring rate.

The roots of the equation for the steady state are

$$\frac{-(r + \mu_L + \alpha - \beta\Pi(L)) \pm \sqrt{(r + \mu_L + \alpha - \beta\Pi(L))^2 + 4((\mu_L + \alpha)\Pi(L) + x_{ss})\beta}}{2\beta}.$$

Let's denote by Π^L_- and Π^L_+ the smaller and larger solution to the quadratic equation (E.5), respectively. We show next that only one of these roots is consistent with $m_\tau \geq 0$.

Claim E.2 (Bad News)**.** *If $\mu_L > \mu_H$ then*

$$\alpha + \beta\Pi^L_+ < 0.$$

Given that we are in the bad news case, $m_\tau > 0$ only if $\Pi_\tau < -\alpha/\beta$. When $\mu_L > \mu_H$, the larger root Π_+^L is

$$\begin{aligned}\Pi_+^L &= \frac{r+\mu_L+\alpha-\beta\Pi(L)+\sqrt{(r+\mu_L+\alpha-\beta\Pi(L))^2-4((\mu_L+\alpha)\Pi(L)+x_{ss})(-\beta)}}{-2\beta}\\ &> \frac{2(r+\mu_L+\alpha-\beta\Pi(L))+2\sqrt{((\mu_L+\alpha)\Pi(L)+x_{ss})(-\beta)}}{-2\beta}\\ &= -\frac{\alpha}{\beta}+\frac{r+\mu_L-\beta\Pi(L)}{-\beta}+\frac{\sqrt{((\mu_L+\alpha)\Pi(L)+x_{ss})(-\beta)}}{-\beta}\\ &> -\frac{\alpha}{\beta}.\end{aligned}$$

Hence, in the bad news case only the trajectory towards the saddle point is consistent with $m_\tau > 0$.

Claim E.3 (Good News)**.** *If* $\mu_L < \mu_H$ *then*

$$\alpha + \beta\Pi_-^L < 0.$$

In the good news case, $m_\tau > 0$ only if $\Pi_\tau > -\alpha/\beta$. The smaller root is

$$\Pi_-^L = \frac{-(r+\mu_L+\alpha-\beta\Pi(L))-\sqrt{(r+\mu_L+\alpha-\beta\Pi(L))^2+4((\mu_L+\alpha)\Pi(L)+x_{ss})\beta}}{2\beta}$$

If $\Pi_-^L \leq 0$ then there is nothing to prove as the payoff of the firm cannot be negative. Accordingly, let's restrict attention to parameters such that $\Pi_-^L > 0$. We have that $\Pi_-^L > 0$ if and only if

$$(r+\mu_L-\beta\Pi(L))+\sqrt{(r+\mu_L+\alpha-\beta\Pi(L))^2+4((\mu_L+\alpha)\Pi(L)+x_{ss})\beta} < -\alpha$$

Monitoring is positive at iff $\Pi_-^L > -\alpha/\beta$ which requires

$$(r+\mu_L-\alpha+\beta\Pi(L))+\sqrt{(r+\mu_L+\alpha-\beta\Pi(L))^2+4((\mu_L+\alpha)\Pi(L)+x_{ss})\beta} < 0$$

We consider two separate cases:

Case $\alpha \leq 0$ Using the condition for $\Pi_-^L > 0$ we get the inequality

$$\begin{aligned}r+\mu_L-\alpha+\beta\Pi(L)+\sqrt{(r+\mu_L+\alpha-\beta\Pi(L))^2+4((\mu_L+\alpha)\Pi(L)+x_{ss})\beta} &>\\ 2(r+\mu_L+\beta\Pi(L))-\alpha+2\sqrt{(r+\mu_L+\alpha-\beta\Pi(L))^2+4((\mu_L+\alpha)\Pi(L)+x_{ss})\beta} &> 0\end{aligned}$$

which contradicts the condition for positive monitoring $\Pi_-^L > -\alpha/\beta$.

Case $\alpha > 0$ If $(r+\mu_L+\alpha-\beta\Pi(L)) > 0$ then we get an immediate contradiction with the hypothesis that $\Pi_-^L > 0$. Hence, assume that $(r+\mu_L+\alpha-\beta\Pi(L)) < 0$. For any $b > 0$ and $a < 0$

we have the following inequality

$$\sqrt{a^2+b} > |a| \Rightarrow -a - \sqrt{a^2+b} < -a - |a| = 0.$$

If $\alpha > 0$ then we have $4((\mu_L + \alpha)\Pi(L) + x_{ss})\beta > 0$. Setting $a = (r + \mu_L + \alpha - \beta\Pi(L)) < 0$ and $b = 4((\mu_L + \alpha)\Pi(L) + x_{ss})\beta > 0$ in the previous inequality we get

$$\Pi_-^L = \frac{-(r + \mu_L + \alpha - \beta\Pi(L)) - \sqrt{(r + \mu_L + \alpha - \beta\Pi(L))^2 + 4((\mu_L + \alpha)\Pi(L) + x_{ss})\beta}}{2\beta} < 0,$$

which yields a contradiction to $\Pi_-^L > 0$. $\square$

F Discrete Time Model

In this appendix we consider a discrete version of the model. We show that the solution in the discrete time version has a similar form to the one in the continuous time model and converges to the continuous time policy when the time between periods goes to zero. Remember that the original continuous time problem is

$$\begin{cases} \sup_F \int_0^\infty \left(\int_0^\tau e^{-rs}u(x_s^\theta)\mathrm{d}s + e^{-r\tau}\mathcal{M}(\mathbf{U}, x_\tau^\theta)\right)\mathrm{d}F(\tau) \\ \text{subject to} \\ \int_\tau^\infty \left(e^{-(r+\lambda)(s-\tau)} - \underline{q}\right)\mathrm{d}F(s) \geq 0. \end{cases}$$

Suppose that the the principal can only monitor at (real) time $\tau \in \{0, \Delta, 2\Delta, \dots\}$. Let $\delta = e^{-r\Delta}$ and $\beta = e^{-\lambda\Delta}$. That is, δ is the one period discount factor and β is the one period transition probability. With some abuse of notation, let's define the utility function in period t to be

$$u(x_t) = \int_0^\Delta e^{-rs}\tilde{u}\left(x_t e^{-\lambda s} + \bar{a}\left(1 - e^{-\lambda s}\right)\right)\mathrm{d}s,$$

where $\tilde{u}$ is the flow payoff in the original continuous time version of the model. Notice that we have that

$$x_{t+1} = (1-\beta)\bar{a} + \beta x_t.$$

Let's denote the realized payoff of monitoring in period t by

$$V_t = \sum_{n=0}^{t} \delta^n u(x_n) + \delta^t \mathcal{M}(\mathbf{U}, x_t),$$

and notice that $V_t = V(t\Delta)$ where

$$V(\tau) = \int_0^\tau e^{-rs}\tilde{u}(x_s^\theta)\mathrm{d}s + e^{-r\tau}\mathcal{M}(\mathbf{U}, x_\tau^\theta)$$

We can now write the discrete time version of the problem as

$$\begin{cases} \max_{p_t} \sum_0^\infty V_t p_t \\ \text{subject to} \\ \sum_{k\geq 0}\left((\beta\delta)^k - \underline{q}\right)p_{t+k} \geq 0, \ \forall t \geq 0 \\ \sum_{t\geq 0} p_t = 1 \end{cases}$$

Let $(\beta\delta)^t\psi_t$ be the Langrange multiplier of the constraint at time t. The Langrangian is

$$L = \sum_{t=0}^{\infty} V_t p_t + \sum_{t=0}^{\infty} \psi_t \sum_{k\geq 0} \left((\beta\delta)^{t+k} - (\beta\delta)^t \underline{q} \right) p_{t+k}$$

We can get replace $p_0 = 1 - \sum_{t\geq 1} p_t$ and write the primal optimization problem as

$$\begin{cases} \max_{p_t} \sum_{t=1}^{\infty} (V_t - V_0) p_t \\ \text{subject to} \\ (1-\underline{q}) + \sum_{t=1}^{\infty} \left((\beta\delta)^t - 1 \right) p_t \geq 0 \\ \sum_{k=0}^{\infty} \left((\beta\delta)^{t+k} - (\beta\delta)^t \underline{q} \right) p_{t+k} \geq 0, \ \forall t \geq 1 \\ \sum_{t=1}^{\infty} p_t \leq 1 \end{cases} \tag{F.1}$$

The Lagrangean for this problem is

$$\mathcal{L}(p,\eta,\psi) = (1-\underline{q})\psi_0 + \eta + \sum_{t=1}^{T} \left(V_t - V_0 - \eta + \left((\beta\delta)^t - 1 \right) \psi_0 + \sum_{k=1}^{t} \psi_k \left((\beta\delta)^t - (\beta\delta)^k \underline{q} \right) \right) p_t$$

From here, we get that for all $t \geq 1$ it must be the case that

$$V_t - V_0 - \eta + \left((\beta\delta)^t - 1 \right) \psi_0 + \sum_{k=1}^{t} \psi_k \left((\beta\delta)^t - (\beta\delta)^k \underline{q} \right) \leq 0,$$

which means that the dual of the optimization problem in (F.1) is

$$\begin{cases} \min \ (1-\underline{q})\psi_0 + \eta \\ \text{subject to} \\ V_t - V_0 - \eta + \left((\beta\delta)^t - 1 \right) \psi_0 + \sum_{k=1}^{t} \psi_k \left((\beta\delta)^t - (\beta\delta)^k \underline{q} \right) \leq 0, \ \forall t \geq 1 \\ \eta \geq 0, \ \psi_t \geq 0 \ \forall t \geq 0 \end{cases} \tag{F.2}$$

We start with the following Lemma characterizing policies that keep the incentive compatibility constraint binding.

Lemma F.1. *Let $\{p_t^{t^*}\}_{t\geq 0}$ be given by*

$$p_t^{t^*} = \begin{cases} 0 & \text{if } t \leq t^* - 1 \\ p_{t^*} & \text{if } t = t^* \\ \left(\frac{1-\underline{q}}{1-\beta\delta\underline{q}} \right)^{t-t^*-1} p_{t^*+1}^{t^*} & \text{if } t \geq t^* + 1, \end{cases}$$

$$p^{t^*}_{t^*} = \frac{\underline{q}(1-(\beta\delta)^{t^*+1})}{(\beta\delta)^{t^*} - \underline{q}(\beta\delta)^{t^*+1}}$$

$$p^{t^*}_{t^*+1} = \frac{\underline{q}(1-\beta\delta)}{1-\beta\delta\underline{q}} \frac{(\beta\delta)^{t^*} - \underline{q}}{(\beta\delta)^{t^*} - \underline{q}(\beta\delta)^{t^*+1}},$$

then the incentive compatibility constraint is binding at $t = 0$ and $t \geq t^* + 1$ and slack for $0 < t \leq t^*$. Alternatively, let $\{p^0_t\}_{t\geq 0}$ be given by

$$p^0_t = \frac{\underline{q}(1-\beta\delta)}{1-\beta\delta\underline{q}} \left(\frac{1-\underline{q}}{1-\beta\delta\underline{q}}\right)^t,$$

then the incentive compatibility constraint is binding at all $t \geq 0$.

Proof. If the incentive compatibility constraint is binding for $t \geq t^* + 1$ then it must be the case that $\sum_{k\geq 0} \left((\beta\delta)^{t+k} - (\beta\delta)^t \underline{q}\right) p_{t+k} = 0$ for all $t \geq t^*+1$. Suppose that $p_t = \alpha^{t-t^*-1} p_{t^*+1}$. Replacing in the incentive compatibility constraint at time $t^* + 1$ we get

$$\sum_{k\geq 0} \left((\beta\delta)^k - \underline{q}\right) \alpha^k = 0$$

which means that

$$\frac{1}{1-\beta\delta\alpha} = \frac{\underline{q}}{1-\alpha},$$

and solving for α we get

$$\alpha = \frac{1-\underline{q}}{1-\beta\delta\underline{q}}.$$

Next, we determine p_{t^*} and p_{t^*+1}. The incentive compatibility at time zero requires that

$$\begin{aligned}
(1-\underline{q}) + \sum_{t=1}^{\infty} \left((\beta\delta)^t - 1\right) p_t &= (1-\underline{q}) + \sum_{t=t^*}^{\infty} \left((\beta\delta)^t - 1\right) p_t \\
&= \sum_{t=t^*}^{\infty} (\beta\delta)^t p_t - \underline{q} \\
&= (\beta\delta)^{t^*} p_{t^*} + \sum_{t=t^*+1}^{\infty} (\beta\delta)^t p_t - \underline{q} \\
&= (\beta\delta)^{t^*} - \underline{q} + \sum_{t=t^*+1}^{\infty} ((\beta\delta)^t - (\beta\delta)^{t^*}) p_t \\
&= 0
\end{aligned}$$

where we have used the fact that $p_{t^*} = 1 - \sum_{t=t^*+1}^{\infty} p_t$. We have that

$$\sum_{t=t^*+1}^{\infty} p_t = p_{t^*+1} \sum_{k=0}^{\infty} \left(\frac{1-\underline{q}}{1-\beta\delta\underline{q}} \right)^k$$
$$= p_{t^*+1} \frac{1-\beta\delta\underline{q}}{\underline{q}(1-\beta\delta)}$$
$$\sum_{t=t^*+1}^{\infty} (\beta\delta)^t p_t = p_{t^*+1} (\beta\delta)^{t^*+1} \sum_{t=t^*+1}^{\infty} \left(\frac{\beta\delta(1-\underline{q})}{1-\beta\delta\underline{q}} \right)^k$$
$$= p_{t^*+1} (\beta\delta)^{t^*+1} \frac{1-\beta\delta\underline{q}}{1-\beta\delta}$$

Moreover, we have that

Replacing in the IC constraint we get

$$p_{t^*+1} = \frac{\underline{q}(1-\beta\delta)}{1-\beta\delta\underline{q}} \frac{(\beta\delta)^{t^*} - \underline{q}}{(\beta\delta)^{t^*} - \underline{q}(\beta\delta)^{t^*+1}}$$

□

We have the following Proposition characterizing the optimal policy

Theorem F.2. *Let* $\bar{t} = \max\{t \geq 0 : (\beta\delta)^t \geq \underline{q}\}$. *If* V_t *has a maximum at* $t \leq \bar{t}$ *then the optimal policy is deterministic monitoring at the maximum. Otherwise, the optimal policy is the following:*

1. *The optimal policy is* p_t^0 *if*

$$V_1 - V_0 \leq \frac{(1-\beta\delta)}{\beta\delta(1-\underline{q})} \sum_{t=1}^{\infty} \left(\frac{p_t^0}{1-p_0^0} V_t - V_1 \right),$$

in which case the incentive compatibility constraint is binding at all times.

2. *The optimal policy is* p_t^1 *if*

$$V_1 - V_0 \geq \frac{1-\beta\delta}{\beta\delta(1-\beta\delta\underline{q})} \left(\sum_{n=1}^{\infty} \frac{p_{1+n}^1}{1-p_1} V_{1+n} - V_1 \right)$$
$$V_2 - V_1 \leq \frac{1-\beta\delta}{1-\beta\delta\underline{q}} \sum_{n=1}^{\infty} \left(\frac{p_{1+n}^1}{1-p_1} V_{1+n} - V_1 \right),$$

3. The optimal policy is $p_t^{t^*}$, $1 < t^* \leq \bar{t}$ *if*

$$V_{t^*} - V_0 \geq \frac{1-(\beta\delta)^{t^*}}{(\beta\delta)^{t^*}(1-\beta\delta\underline{q})}\left(\sum_{n=1}^{\infty}\frac{p_{t^*+n}^{t^*}}{1-p_{t^*}^{t^*}}V_{t^*+n} - V_{t^*}\right)$$

$$V_{t^*} - V_{t^*-1} \geq \frac{1-\beta\delta}{1-\beta\delta\underline{q}}\sum_{n=1}^{\infty}\left(\frac{p_{t^*-1+n}^{t^*}}{1-p_{t^*-1}^{t^*}}V_{t^*-1+n} - V_{t^*}\right)$$

$$V_{t^*+1} - V_{t^*} \leq \frac{1-\beta\delta}{1-\beta\delta\underline{q}}\sum_{n=1}^{\infty}\left(\frac{p_{t^*+n}^{t^*}}{1-p_{t^*}^{t^*}}V_{t^*+n} - V_{t^*}\right),$$

4. The optimal policy is

$$p_t = \begin{cases} 0 & \text{if } t < \bar{t} \\ \frac{\underline{q}-(\beta\delta)^{\bar{t}+1}}{(\beta\delta)^{\bar{t}}(1-\beta\delta)} & \text{if } t = \bar{t} \\ \frac{(\beta\delta)^{\bar{t}}-\underline{q}}{(\beta\delta)^{\bar{t}}(1-\beta\delta)} & \text{if } t = \bar{t}+1 \\ 0 & \text{if } t > \bar{t}+1, \end{cases}$$

if for all $t \leq \bar{t}$

$$V_{t+1} - V_t > \frac{1-\beta\delta}{1-\beta\delta\underline{q}}\sum_{n=1}^{\infty}\left(\frac{p_{t+n}^{t}}{1-p_t^t}V_{t+n} - V_t\right)$$

Proposition F.3. *In the limit, when* $\Delta \to 0$, *the optimal policy converges to the one in the continuous time model.*

Proof. First, take the limit of the policy in Lemma F.1. First, we consider the case with $t^* \geq 1$. Let $\tau = t\Delta$ and replace the expressions for δ and β. First, we get that

$$p_{\tau*/\Delta}^{\tau*/\Delta} = \frac{\underline{q}(1-e^{-(r+\lambda)(\tau^*+\Delta)})}{e^{-(r+\lambda)\tau^*} - \underline{q}e^{-(r+\lambda)(\tau^*+\Delta)}}$$

$$p_{\tau*/\Delta+1}^{\tau*/\Delta} = \frac{\underline{q}(1-e^{-(r+\lambda)\Delta})}{1-e^{-(r+\lambda)\Delta}\underline{q}}\frac{e^{-(r+\lambda)\tau^*} - \underline{q}}{e^{-(r+\lambda)\tau^*} - \underline{q}e^{-(r+\lambda)(\tau^*+\Delta)}},$$

so taking the limit when $\Delta \to 0$ we get

$$\lim_{\Delta\to 0} p_{\tau*/\Delta}^{\tau*/\Delta} = \frac{\underline{q}(e^{(r+\lambda)\tau^*}-1)}{1-\underline{q}}$$

$$\lim_{\Delta\to 0} p_{\tau*/\Delta+1}^{\tau*/\Delta} = 0.$$

Second, we get that for $\tau \geq \tau^*$

$$p_{\tau/\Delta}^{\tau*/\Delta} = \left(\frac{1-e^{-(r+\lambda)\Delta}\underline{q}}{1-\underline{q}}\right)^{-(\tau-\tau^*)/\Delta}\frac{1-e^{-(r+\lambda)\Delta}\underline{q}}{1-\underline{q}}p_{\tau*/\Delta+1}^{\tau*/\Delta}$$

Notice that

$$\left(\frac{1-e^{-(r+\lambda)\Delta}\underline{q}}{1-\underline{q}}\right)^{-(\tau-\tau^*)/\Delta}\frac{1-e^{-(r+\lambda)\Delta}\underline{q}}{1-\underline{q}}p^{t^*}_{t^*+1} =$$

$$\left(1+\frac{\underline{q}}{1-\underline{q}}\frac{1-e^{-(r+\lambda)\Delta}}{\Delta}\Delta\right)^{-(\tau-\tau^*)/\Delta}\frac{\underline{q}(1-e^{-(r+\lambda)\Delta})}{(1-\underline{q})\Delta}\frac{1-e^{(r+\lambda)\tau^*}\underline{q}}{1-\underline{q}e^{-(r+\lambda)\Delta}}\Delta,$$

so taking the limit we get that for any $t \geq t^*+1$ with $\tau = t\Delta$

$$\lim_{\Delta\to 0}\sum_{n\geq \tau/\Delta} p^{\tau*/\Delta}_{n/\Delta} = \frac{1-e^{(r+\lambda)\tau^*}\underline{q}}{1-\underline{q}}\int_{\tau}^{\infty} e^{-m^*(s-\tau^*)}m^*\mathrm{d}\tau = \frac{1-e^{(r+\lambda)\tau^*}\underline{q}}{1-\underline{q}}e^{-m^*(\tau-\tau^*)}$$

which verifies that the policy converges to the on in the continuous time model. Similarly, when $t^*=0$ the policy is given by

$$p^0_{\tau/\Delta} = \frac{\underline{q}(1-e^{-(r+\lambda)\Delta})}{1-e^{-(r+\lambda)\Delta}\underline{q}}\left(\frac{1-e^{-(r+\lambda)\Delta}\underline{q}}{1-\underline{q}}\right)^{-\tau/\Delta}$$

so for $\tau = t\Delta$ we get

$$\lim_{\Delta\to 0}\sum_{n\geq t} p^0_n = \int_{\tau}^{\infty} e^{-m^*(s-\tau^*)}m^*\mathrm{d}\tau = e^{-m^*(\tau-\tau^*)}$$

Finally, notice that for the final case in Theorem F.2 we have that $p_{\bar{t}}+p_{\bar{t}+1}=1$, $\bar{\tau}=\Delta\bar{t}\to\frac{1}{r+\lambda}\log\frac{1}{\underline{q}}$ so we have monitoring with probability 1 at time $\bar{\tau}$. Finally, we can verify that the condition

$$V_{\tau*/\Delta}-V_{\tau*/\Delta-1} \geq \frac{1-\beta\delta}{1-\beta\delta\underline{q}}\sum_{n=1}^{\infty}\left(\frac{p^{\tau*/\Delta}_{\tau*/\Delta-1+n}}{1-p^{\tau*/\Delta}_{\tau*/\Delta-1}}V_{\tau*/\Delta-1+n}-V_{\tau*/\Delta}\right)$$

$$V_{\tau*/\Delta+1}-V_{\tau*/\Delta} \leq \frac{1-\beta\delta}{1-\beta\delta\underline{q}}\sum_{n=1}^{\infty}\left(\frac{p^{\tau*/\Delta}_{\tau*/\Delta+n}}{1-p^{\tau*/\Delta}_{\tau*/\Delta}}V_{\tau*/\Delta+n}-V_{\tau*/\Delta}\right),$$

converges to

$$V'(\tau^*) = \frac{r+\lambda}{1-\underline{q}}\left(E[V(\tau)|\tau>\tau^*]-V(\tau^*)\right).$$

□

Proof Theorem F.2

The case in which V_t reaches a maximum for some $t \leq \bar{t}$ is trivial as in this case all the constraints are slack, so we only need to consider the case in which V_t is increasing for $t \leq \bar{t}$. In this case, the proof relies on the theory of weak duality for infinite-dimensional linear programming problems (Anderson and Nash, 1987, Theorem 2.1). The policy in Theorem F.2 is feasible for the primal problem. We conjecture that the optimal policy takes this form, and construct Lagrange multipliers that are dual feasible. By weak duality, the value of the dual given these Lagrange multipliers provides an upper bound on the value of the primal. We then verify that the value of the dual corresponds to the value of the primal given our conjectured policy, which establishes the optimality of our conjectured policy.

We start with some Lemmas that will be useful in the computations.

Lemma F.4. *If u is convex then $u(x_{t+1}) - \beta u(x_t)$ is increasing.*

Proof. Since $\beta > 0$, the inequality we are claiming is equivalent to

$$u\left(x_t\right) \leq \frac{1}{1+\beta} u\left(x_{t+1}\right) + \frac{\beta}{1+\beta} u\left(x_{t-1}\right). \tag{F.3}$$

Now, using twice the definition of x_t, we have:

$$\frac{1}{1+\beta} x_{t+1} + \frac{\beta}{1+\beta} x_{t-1} = \frac{\beta x_t + (1-\beta)\,\overline{a} + \beta x_{t-1}}{1-\beta} = x_t$$

In other words, x_t is a weighted average of x_{t+1} and x_{t-1}. Note that the weights are the same as in $(F.3)$ and hence convexity of u implies that $(F.3)$ indeed holds. ☐

Lemma F.5.

$$\mathcal{M}(\mathbf{U}, x_t) - \beta \mathcal{M}(\mathbf{U}, x_{t-1}) = (1-\beta)\mathcal{M}(\mathbf{U}, \bar{a})$$

F.1 Construction of the Lagrange Multipliers

Let t^* be the the first date at which there is monitoring with positive probability. The objective is to construct multipliers when the incentive compatibility constraint is binding after $t^* + 1$. Let's define

$$F_t \equiv V_t - V_0 - \eta + \left((\beta\delta)^t - 1\right)\psi_0 + \sum_{k=1}^{t} \psi_k \left((\beta\delta)^t - (\beta\delta)^k \underline{q}\right),$$

which correspond to the LHS of the constraint at time t in the dual problem. Clearly, the multipliers (η, ψ) are dual feasible if and only if $F_t \geq 0$. Taking the difference $F_{t+1} - F_t$ we get

$$F_{t+1} - F_t = V_{t+1} - V_t + (1-\underline{q})(\beta\delta)^{t+1}\psi_{t+1} - \left((\beta\delta)^t - (\beta\delta)^{t+1}\right)\sum_{k=0}^{t}\psi_k$$

where

$$V_{t+1} - V_t = \delta^{t+1} u(x_{t+1}) + \delta^{t+1}\mathcal{M}(\mathbf{U}, x_{t+1}) - \delta^t \mathcal{M}(\mathbf{U}, x_t)$$

If $F_t = 0$ and $\psi_{t+1} > 0$, then it must be the case that $F_{t+1} = 0$, so $F_{t+1} - F_t = 0$. This requires that

$$(1-\underline{q})(\beta\delta)^{t+1}\psi_{t+1} = -(V_{t+1} - V_t) + \left((\beta\delta)^t - (\beta\delta)^{t+1}\right)\sum_{k=0}^{t}\psi_k > 0 \tag{F.4}$$

We want to derive a difference equation for $\tilde{\psi}_t = \beta^t \psi_t$ such that $F_t = 0$ for all $t \geq t^* + 1$. From equation (F.4) we have that if $F_{t+1} = 0$, then ψ_{t+1} is given by

$$(1-\underline{q})(\beta\delta)^{t+1}\psi_{t+1} = -(V_{t+1} - V_t) + \left((\beta\delta)^t - (\beta\delta)^{t+1}\right)\sum_{k=0}^{t}\psi_k \tag{F.5}$$

Considering equation at time t and multiplying both sides by $\beta\delta$ we get

$$(1-\underline{q})(\beta\delta)^{t+1}\psi_t = -\beta\delta(V_t - V_{t-1}) + \left((\beta\delta)^t - (\beta\delta)^{t+1}\right)\sum_{k=0}^{t-1}\psi_k \tag{F.6}$$

Taking the difference between (F.1) and (F.4), and using the definition $\tilde{\psi}_t = \beta^t \psi_t$ we get

$$(1-\underline{q})\delta^{t+1}\tilde{\psi}_{t+1} = \beta\delta(V_t - V_{t-1}) - (V_{t+1} - V_t) + \left(1 - \underline{q}\beta\delta\right)\delta^t\tilde{\psi}_t$$

Hence, for any $t > t^* + 1$, $\tilde{\psi}_t$ satisfies the recursion

$$\tilde{\psi}_{t+1} = \alpha\tilde{\psi}_t - h_t \tag{F.7}$$

where

$$\alpha \equiv \frac{1-\underline{q}\beta\delta}{(1-\underline{q})\delta}$$
$$h_t \equiv \frac{1}{1-\underline{q}}\left(u(x_{t+1}) - \beta u(x_t) + (1-\beta)\mathcal{M}(\mathbf{U}, \bar{a})\right).$$

and $\alpha > 1$, h_t is increasing by Lemma F.4. Solving recursively, we get that

$$\tilde{\psi}_{t^*+k+1} = \alpha^k\tilde{\psi}_{t^*+1} - \sum_{n=0}^{k-1}\alpha^{k-1-n}h_{t^*+n+1},\ k \geq 1. \tag{F.8}$$

The final step is to specify conditions on the initial value ψ_{t^*+1} that guarantee that the sequence $\{\psi_t\}_{t\geq t^*+1}$ in equation (F.8) is nonnegative.

Lemma F.6. *Fix* $0 \leq t^* \leq \bar{t}$ *then*

1. *If* $u(x_{t^*+1}) - \beta u(x_{t^*}) + (1-\beta)\mathcal{M}(U, \bar{a}) \geq 0$ *then* $\psi_t \geq 0$ *for all* $t \geq t^* + 1$ *if and only if*

$$\psi_{t^*+1} \geq \frac{1}{\beta^{t^*+1}(1-\underline{q})}\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1}\left(u(x_{t^*+n+2}) - \beta u(x_{t^*+1+n}) + (1-\beta)\mathcal{M}(U, \bar{a})\right)$$

2. *If* $u(x_{t^*+1}) - \beta u(x_{t^*}) + (1-\beta)\mathcal{M}(U, \bar{a}) < 0$ *then* $\psi_t \geq 0$ *for all* $t \geq t^* + 1$ *if and only if* $\psi_{t^*+1} \geq 0$.

Proof. Consider an initial condition of the form

$$\tilde{\psi}_{t^*+1} = \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n} + \Delta \tag{F.9}$$

for some Δ to be determined. Replacing the initial condition in (F.9) in (F.8) above, we get

$$\begin{aligned}\tilde{\psi}_{t^*+1+k} &= \alpha^k \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n} + \alpha^k \Delta - \sum_{n=0}^{k-1} \alpha^{k-1-n} h_{t^*+1+n} \\ &= \sum_{n=k}^{\infty} \alpha^{k-1-n} h_{t^*+1+n} + \alpha^k \Delta \\ &= \alpha^k \left(\sum_{n=k}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n} + \Delta \right)\end{aligned}$$

Hence, $\tilde{\psi}_{t^*+1+n}$ is nonnegative if

$$\Delta \geq - \sum_{n=k}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n}, \forall k \geq 0. \tag{F.10}$$

Let's define

$$H_k \equiv \sum_{n=k}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n},$$

where h_t is increasing.

First, let's consider the case with $u(x_{t^*+1}) - \beta u(x_{t^*}) + (1-\beta)\mathcal{M}(U, \bar{a}) \geq 0$. Because h_{t^*+1+n} is increasing, we have in this case that h_{t^*+1+n} is positive for all n, so we need to take $\Delta = 0$, which corresponds to the condition in the Lemma.

Next, we consider the case with $u(x_{t^*+1}) - \beta u(x_{t^*}) + (1-\beta)\mathcal{M}(U, \bar{a}) < 0$. Let's define $t^\dagger = \sup\{t > t^* + 1 : h_t \leq 0\}$. By definition, H_k is decreasing for $k < t^\dagger - t^* - 1$ and increasing for $k \geq t^\dagger - t^* - 1$. This means that H_k has a minimum at $k^\dagger \equiv t^\dagger - t^* - 1$, and so $-H_k$ has a maximum at $k^\dagger$. Thus, $\tilde{\psi}_{t^*+1+k}$ is nonnegative if and if

$$\Delta \geq - \sum_{n=k^\dagger}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n}.$$

Replacing in (F.9) we get that

$$\begin{aligned}\tilde{\psi}_{t^*+1} &\geq \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n} - \sum_{n=k^\dagger}^{\infty} \frac{1}{\alpha^{n+1}} h_{t^*+1+n} \\ &= \sum_{n=0}^{k^\dagger-1} \frac{1}{\alpha^{n+1}} h_{t^*+1+n}.\end{aligned}$$

By definition of $k^\dagger$, h_{t^*+1+n} for all $n \leq k^\dagger - 1$ which means that $\sum_{n=0}^{k^\dagger-1} \frac{1}{\alpha^{n+1}} h_{t^*+1+n} \leq 0$, so it is enough to consider $\tilde{\psi}_{t^*+1} \geq 0$ □

F.2 Verification of Optimal Policy

F.2.1 Case: $0 < t^* < \bar{t}$

First, we consider a policy in which $0 < t^* < \bar{t}$. The following propositions characterizes sufficient conditions for this policy to be optimal.

Proposition F.7. *Let $\bar{t} = \max\{t \geq 0 : (\beta\delta)^t \geq \underline{q}\}$. Let $0 < t^* \leq \bar{t}$ be such that: if $t^* = 1$ then*

$$V_{t^*} - V_0 \geq \frac{1-(\beta\delta)^{t^*}}{(\beta\delta)^{t^*}(1-\beta\delta\underline{q})} \left(\sum_{n=1}^{\infty} \frac{p^1_{t^*+n}}{1-p^1_{t^*}} V_{t^*+n} - V_{t^*} \right)$$
$$V_{t^*+1} - V_{t^*} \leq \frac{1-\beta\delta}{1-\beta\delta\underline{q}} \sum_{n=1}^{\infty} \left(\frac{p^1_{t^*+n}}{1-p^1_{t^*}} V_{t^*+n} - V_{t^*} \right),$$

and if $t^* > 1$ then

$$V_{t^*} - V_0 \geq \frac{1-(\beta\delta)^{t^*}}{(\beta\delta)^{t^*}(1-\beta\delta\underline{q})} \left(\sum_{n=1}^{\infty} \frac{p^{t^*}_{t^*+n}}{1-p^{t^*}_{t^*}} V_{t^*+n} - V_{t^*} \right)$$
$$0 \geq u(x_{t^*}) - \beta u(x_{t^*-1}) + (1-\beta)\mathcal{M}(U, \bar{a})$$
$$V_{t^*+1} - V_{t^*} \leq \frac{1-\beta\delta}{1-\beta\delta\underline{q}} \sum_{n=1}^{\infty} \left(\frac{p^{t^*}_{t^*+n}}{1-p^{t^*}_{t^*}} V_{t^*+n} - V_{t^*} \right).$$

If this conditions are satisfied, then the optimal policy is $\{p^{t^*}_t\}_{t\geq 0}$.

Proof. The first condition at time t^* yields

$$V_{t^*} - V_0 + \left((\beta\delta)^{t^*} - 1 \right) \psi_0 = \eta.$$

Replacing η in the first order condition at time t^*+1 yields

$$V_{t^*+1} - V_{t^*} - \left((\beta\delta)^{t^*} - (\beta\delta)^{t^*+1} \right) \psi_0 + (\beta\delta)^{t^*+1}(1-\underline{q})\psi_{t^*+1} = 0.$$

Solving for ψ_0 and η we get

$$\psi_0 = \frac{V_{t^*+1} - V_{t^*}}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} + \frac{\beta\delta(1-\underline{q})}{1-\beta\delta} \psi_{t^*+1} \tag{F.11}$$
$$\eta = \frac{V_{t^*} - V_{t^*+1} + (\beta\delta)^{t^*}(V_{t^*+1} - \beta\delta V_{t^*})}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} - V_0 - \frac{\beta\delta(1-(\beta\delta)^{t^*})}{1-\beta\delta}(1-\underline{q})\psi_{t^*+1} \tag{F.12}$$

From here we get that

$$(1-\underline{q})\psi_0+\eta=\frac{\left((\beta\delta)^{t^*}-\underline{q}\right)V_{t^*+1}-\left((\beta\delta)^{t^*+1}-\underline{q}\right)V_{t^*}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}-V_0+\frac{\beta\delta((\beta\delta)^{t^*}-\underline{q})}{1-\beta\delta}(1-\underline{q})\psi_{t^*+1} \quad \text{(F.13)}$$

Notice that $t^*\leq\min\{t>0:(\beta\delta)^t\geq\underline{q}\}$, which means that $1+\frac{(\beta\delta)^{t^*+1}-\underline{q}}{1-\beta\delta}>0$ so the solution always involves choosing the smallest possible ψ_{t^*+1}. Suppose that $t^*\geq 1$ satisfies the conditions in the proposition and consider the multiplier

$$\eta=\frac{V_{t^*}-V_{t^*+1}+(\beta\delta)^{t^*}\left(V_{t^*+1}-\beta\delta V_{t^*}\right)}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}-V_0-\frac{\beta\delta(1-(\beta\delta)^{t^*})}{1-\beta\delta}(1-\underline{q})\psi_{t^*+1}$$

$$\psi_0=\frac{V_{t^*+1}-V_{t^*}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}+\frac{\beta\delta(1-\underline{q})}{1-\beta\delta}\psi_{t^*+1}$$

$$\psi_t=0,\ 0<t\leq t^*$$

$$\psi_t=\frac{1}{\beta^t(1-\underline{q})}\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1}\left(u(x_{t+n+1})-\beta u(x_{t+n})+(1-\beta)\mathcal{M}(U,\bar{a})\right),\ t\geq t^*+1$$

By Lemma F.6 , the multipliers $\{\psi_t\}_{t\geq 0}$ are nonnegative. By weak duality, to verify the optimality of $\{p_t^*\}_{t\geq 0}$, it is enough to verify that the proposed multipliers are dual feasible and that $(1-\underline{q})\psi_0+\eta$ equals the expected payoff of $\{p_t^{t^*}\}_{t\geq 0}$.

Step 1: First, we verify that $(1-\underline{q})\psi_0+\eta=\sum_{t=0}^{\infty}p_t^*V_t-V_0$. Replacing the expressions for η and ψ_0 we get

$$\begin{aligned}(1-\underline{q})\psi_0+\eta=&\frac{\left((\beta\delta)^{t^*}-\underline{q}\right)V_{t^*+1}-\left((\beta\delta)^{t^*+1}-\underline{q}\right)V_{t^*}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}-V_0\\&+\frac{(\beta\delta)^{t^*}-\underline{q}}{1-\beta\delta}\frac{1}{(\beta\delta)^{t^*}}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}\delta^{t^*+n+2}\left(u(x_{t^*+n+2})-\beta u(x_{t^*+n+1})+(1-\beta)\mathcal{M}(U,\bar{a})\right)\end{aligned} \quad \text{(F.14)}$$

which can be written as

$$\begin{aligned}(1-\underline{q})\psi_0+\eta=&\frac{\left((\beta\delta)^{t^*}-\underline{q}\right)V_{t^*+1}-\left((\beta\delta)^{t^*+1}-\underline{q}\right)V_{t^*}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}-V_0\\&+\frac{(\beta\delta)^{t^*}-\underline{q}}{(\delta\beta)^{t^*}-(\delta\beta)^{t^*+1}}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}\left(V_{t^*+n+2}-(1+\beta\delta)V_{t^*+n+1}+\beta\delta V_{t^*+n}\right)\end{aligned} \quad \text{(F.15)}$$

Using telescopic sums, we have that

$$\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}\left(V_{t^*+n+2}-(1+\beta\delta)V_{t^*+n+1}+\beta\delta V_{t^*+n}\right)=$$

$$\sum_{n=0}^{\infty}\left[\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}V_{t^*+n+2}-\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}\right]-\sum_{n=0}^{\infty}\left[\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}-\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}\right]V_{t^*+n+1}$$

$$-\beta\delta\sum_{n=0}^{\infty}\left[\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}V_{t^*+n+1}-\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n}\right]+\beta\delta\sum_{n=0}^{\infty}\left[\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1}-\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}\right]V_{t^*+n}=$$

$$\beta\delta V_{t^*}-V_{t^*+1}+\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}-\beta\delta\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n}=$$

$$\beta\delta\left(1-\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\right)V_{t^*}-V_{t^*+1}-\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\left(\frac{\beta\delta(1-\underline{q})}{1-\underline{q}\beta\delta}-1\right)\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}=$$

$$\frac{\beta\delta(1-\underline{q})}{1-\underline{q}\beta\delta}V_{t^*}-V_{t^*+1}+\frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}$$

Replacing in equation (F.15) we get

$$(1-\underline{q})\psi_0+\eta=\frac{\left((\beta\delta)^{t^*}-\underline{q}\right)V_{t^*+1}-\left((\beta\delta)^{t^*+1}-\underline{q}\right)V_{t^*}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}-V_0$$

$$+\frac{(\beta\delta)^{t^*}-\underline{q}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}\left(\frac{\beta\delta(1-\underline{q})}{1-\underline{q}\beta\delta}V_{t^*}-V_{t^*+1}+\frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}\right)=$$

$$\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\frac{1-(\beta\delta)^{t^*+1}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}V_{t^*}-V_0+\frac{(\beta\delta)^{t^*}-\underline{q}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}\frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}.$$

So, after some manipulations and replacing $p_t^{t^*}$ we get that

$$\begin{aligned}(1-\underline{q})\psi_0+\eta&=\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\frac{1-(\beta\delta)^{t^*+1}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}V_{t^*}-V_0+\frac{(\beta\delta)^{t^*}-\underline{q}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}\frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}\\
&=\frac{\underline{q}(1-(\beta\delta)^{t^*+1})}{(\beta\delta)^{t^*}-\underline{q}(\beta\delta)^{t^*+1}}V_{t^*}-V_0+\frac{(\beta\delta)^{t^*}-\underline{q}}{(\beta\delta)^{t^*}-\underline{q}(\beta\delta)^{t^*+1}}\frac{\underline{q}(1-\beta\delta)}{1-\underline{q}\beta\delta}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}\\
&=p_{t^*}^{t^*}V_{t^*}-V_0+\sum_{n=0}^{\infty}p_{t^*+1}^{t^*}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{t^*+n+1}\\
&=\sum_{n=0}^{\infty}p_{t^*+n}^{t^*}V_{t^*+n}-V_0=\sum_{t=0}^{\infty}p_t^{*}V_t-V_0\end{aligned}$$

Step 2: The only step left is to verify that $\eta \geq 0$ and $\psi_{t^*+1} \geq 0$. We can write

$$\begin{aligned}\psi_0 &= \frac{V_{t^*+1} - V_{t^*}}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \\ &+ \frac{1}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \sum_{n=0}^{\infty} \left(\frac{(1-\underline{q})}{1-\underline{q}\beta\delta}\right)^{n+1} \delta^{t^*+n+2} \left(u(x_{t^*+n+2}) - \beta u(x_{t^*+n+1}) + (1-\beta)\mathcal{M}(U,\bar{a})\right) \\ &= -\frac{1}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \left(\frac{1-\beta\delta}{1-\underline{q}\beta\delta}\right) V_{t^*} + \frac{1}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2} \sum_{n=0}^{\infty} \left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^n V_{t^*+n+1},\end{aligned}$$

which means that

$$(1-\underline{q})\psi_0 = -\frac{1-\underline{q}}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \left(\frac{1-\beta\delta}{1-\underline{q}\beta\delta}\right) V_{t^*} + \frac{1-\underline{q}}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2} \sum_{n=0}^{\infty} \left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^n V_{t^*+n+1}$$

Using the definition for $p_t^{t^*}$ we get

$$\begin{aligned}\frac{1-\underline{q}}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \left(\frac{1-\beta\delta}{1-\underline{q}\beta\delta}\right) \frac{1}{p_{t^*}^{t^*}} &= \frac{1-\underline{q}}{\underline{q}(1-(\beta\delta)^{t^*+1})} \\ \frac{1-\underline{q}}{(\beta\delta)^{t^*} - (\beta\delta)^{t^*+1}} \frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2} \frac{1}{p_{t^*+1}^{t^*}} &= \frac{1-\underline{q}}{(\beta\delta)^{t^*} - \underline{q}},\end{aligned}$$

so

$$(1-\underline{q})\psi_0 = -\frac{1-\underline{q}}{\underline{q}(1-(\beta\delta)^{t^*+1})} p_{t^*}^{t^*} V_{t^*} + \frac{1-\underline{q}}{(\beta\delta)^{t^*} - \underline{q}} \sum_{n=0}^{\infty} p_{t^*+1+n}^{t^*} V_{t^*+n+1}$$

Hence, replacing in $(1-\underline{q})\psi_0 + \eta = \sum_{t=0}^{\infty} p_t^* V_t - V_0$ (Step 1) we find that

$$\begin{aligned}\eta &= \frac{1-\underline{q}(\beta\delta)^{t^*+1}}{\underline{q}(1-(\beta\delta)^{t^*+1})} p_{t^*}^{t^*} V_{t^*} - V_0 - \frac{1-(\beta\delta)^{t^*}}{(\beta\delta)^{t^*} - \underline{q}(\beta\delta)^{t^*+1}} \frac{(\beta\delta)^{t^*} - \underline{q}(\beta\delta)^{t^*+1}}{(\beta\delta)^{t^*} - \underline{q}} \sum_{n=0}^{\infty} p_{t^*+1+n}^{t^*} V_{t^*+n+1} \\ &= V_{t^*} - V_0 - \frac{1-(\beta\delta)^{t^*}}{(\beta\delta)^{t^*} - \underline{q}(\beta\delta)^{t^*+1}} \left(\sum_{n=1}^{\infty} \frac{p_{t^*+n}^{t^*}}{1-p_{t^*}^{t^*}} V_{t^*+n} - V_{t^*}\right) \\ &\geq 0\end{aligned}$$

In the case of ψ_{t^*+1}, notice that

$$0 \leq \frac{1}{\beta^{t^*+1}(1-\underline{q})} \sum_{n=0}^{\infty} \left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1} \left(u(x_{t^*+n+2}) - \beta u(x_{t^*+1+n}) + (1-\beta)\mathcal{M}(U,\bar{a})\right)$$

if and only if

$$\begin{aligned}
0 &\leq \sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})}{1-\underline{q}\beta}\right)^{n+1}\delta^{t^*+n+2}\left(u(x_{t^*+n+2})-\beta u(x_{t^*+1+n})+(1-\beta)\mathcal{M}(U,\bar{a})\right)\\
&= \frac{\beta\delta(1-\underline{q})}{1-\underline{q}\beta\delta}V_{t^*}-V_{t^*+1}+\frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^n V_{t^*+n+1}\\
&\propto V_{t^*}-V_{t^*+1}+\frac{1-\beta\delta}{1-\underline{q}\beta\delta}\sum_{n=1}^{\infty}\left(\frac{p^{t^*}_{t^*+n}}{1-p^{t^*}_{t^*}}V_{t^*+n}-V_{t^*}\right)
\end{aligned}$$

Step 3: Feasibility for $t<t^*$. If $t^*>1$, then we We need to verify that

$$V_t-V_0-\eta+\left((\beta\delta)^t-1\right)\psi_0\leq 0,\ \forall t<t^*.$$

If we replace η and ψ_0 we get

$$V_t-V_{t^*}+\frac{(\beta\delta)^t-(\beta\delta)^{t^*}}{(\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}}(V_{t^*+1}-V_{t^*})+\frac{(1-\underline{q})\left((\beta\delta)^{t+1}-(\beta\delta)^{t^*+1}\right)}{1-\beta\delta}\psi_{t^*+1}\leq 0 \tag{F.16}$$

Evaluating (F.16) at time t^*-1 and simplifying we get

$$\beta\delta(V_{t^*-1}-V_{t^*})+V_{t^*+1}-V_{t^*}+\frac{\beta\delta(1-\underline{q})\left((\beta\delta)^{t^*}-(\beta\delta)^{t^*+1}\right)}{1-\beta\delta}\psi_{t^*+1}\leq 0.$$

If we replace

$$\beta\delta(V_{t^*-1}-V_{t^*})+V_{t^*+1}-V_{t^*}=\delta^{t^*+1+n}\left(u(x_{t^*+n+1})-\beta u(x_{t^*+n})+(1-\beta)\mathcal{M}(U,\bar{a})\right)$$

and ψ_{t^*+1} we get

$$\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^n\left(u(x_{t^*+n+1})-\beta u(x_{t^*+n})+(1-\beta)\mathcal{M}(U,\bar{a})\right)\leq 0$$

Feasibility for $t<t^*$ follows from the following lemma

Lemma F.8. *If $u(x_{t^*})-\beta u(x_{t^*-1})+(1-\beta)\mathcal{M}(U,\bar{a})\leq 0$ then*

$$V_t-V_0-\eta+\left((\beta\delta)^t-1\right)\psi_0\leq 0,\ \forall t<t^*.$$

Proof. We prove the statement by induction. Let

$$F_t=V_t-V_0-\eta+\left((\beta\delta)^t-1\right)\psi_0$$

an consider periods $t+1$, t and $t-1$, for $1 \leq t \leq t^* - 1$. Then, we get

$$\begin{aligned}\beta\delta(F_t - F_{t-1}) &= \beta\delta(V_t - V_{t-1}) + \left((\beta\delta)^{t+1} - (\beta\delta)^t\right)\psi_0 \\ (F_{t+1} - F_t) &= \beta\delta(V_{t+1} - V_t) + \left((\beta\delta)^{t+1} - (\beta\delta)^t\right)\psi_0,\end{aligned}$$

so taking difference we get

$$\begin{aligned}\beta\delta(F_t - F_{t-1}) - (F_{t+1} - F_t) &= \beta\delta(V_t - V_{t-1}) - (V_{t+1} - V_t) \\ &= -\left(u(x_{t+1}) - \beta u(x_t) + (1-\beta)\mathcal{M}(U, \bar{a})\right)\end{aligned}$$

Given that $u(x_{t+1}) - \beta u(x_t) + (1-\beta)\mathcal{M}(U, \bar{a}) \leq 0$, we get that

$$\beta\delta(F_t - F_{t-1}) \geq (F_{t+1} - F_t) \geq 0$$

which means that

$$F_{t-1} \leq F_t.$$

The results follows from the fact that $F_{t^*} = 0$ and that $u(x_{t+1}) - \beta u(x_t) + (1-\beta)\mathcal{M}(U, \bar{a})$ is increasing. □

□

Notice that

$$\psi_{t^*+1} = \frac{1}{\beta^{t^*+1}(1-\underline{q})} \sum_{n=0}^{\infty} \left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1} \left(u(x_{t^*+n+2}) - \beta u(x_{t^*+1+n}) + (1-\beta)\mathcal{M}(U, \bar{a})\right) \geq 0$$

Suppose that

$$0 \geq \sum_{n=0}^{\infty} \left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1} \left(u(x_{t^*+n+1}) - \beta u(x_{t^*+n}) + (1-\beta)\mathcal{M}(U, \bar{a})\right)$$

which is equivalent to

$$0 \geq \sum_{n=0}^{\infty} \left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n} \left(u(x_{t^*+n+1}) - \beta u(x_{t^*+n}) + (1-\beta)\mathcal{M}(U, \bar{a})\right).$$

If this conditions are satisfied we get that

$$\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1}(u(x_{t^*+n+1})-\beta u(x_{t^*+n})+(1-\beta)\mathcal{M}(U,\bar{a}))$$
$$\geq\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n}(u(x_{t^*+n+1})-\beta u(x_{t^*+n})+(1-\beta)\mathcal{M}(U,\bar{a}))\,,$$

which is equivalent to

$$0\geq u(x_{t^*+1})-\beta u(x_{t^*})+(1-\beta)\mathcal{M}(U,\bar{a}).$$

By Lemma F.4, the right hand side is monotonic, and this means that

$$0\geq u(x_{t^*})-\beta u(x_{t^*-1})+(1-\beta)\mathcal{M}(U,\bar{a}).$$

Moreover, following similar computations as the ones we did to compute $(1-\underline{q})\psi_0+\eta$ we get that

$$0\geq\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1}(u(x_{t^*+n+1})-\beta u(x_{t^*+n})+(1-\beta)\mathcal{M}(U,\bar{a}))$$

is equivalent to

$$V_{t^*}-V_{t^*-1}\geq\frac{1-\beta\delta}{1-\beta\delta\underline{q}}\sum_{n=1}^{\infty}\left(\frac{p^{t^*}_{t^*-1+n}}{1-p^{t^*}_{t^*}}V_{t^*-1+n}-V_{t^*}\right)$$

F.2.2 Case: $p_0>0$

Next, we consider the case where $p_0>0$. In this case, $\sum_{t\geq 1}p_t<1$ so $\eta=0$.

Proposition F.9. *Suppose that*

$$V_1-V_0\leq\frac{(1-\beta\delta)}{\beta\delta(1-\underline{q})}\sum_{t=1}^{\infty}\left(\frac{p^0_t}{1-p^0_0}V_t-V_1\right)$$

then the optimal policy is $\{p^0_t\}_{t\geq 0}$

Proof. Because $p_0>0$, the constraint $\sum_{t\geq 1}p_t\leq 1$ is slack and $\eta=0$. Consider the multiplier at $t=1$

$$\psi_1=\frac{1}{\beta\delta(1-\underline{q})}\sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1}\delta^{n+2}(u(x_{n+2})-\beta u(x_{n+1})+(1-\beta)\mathcal{M}(U,\bar{a}))$$
$$=\frac{1}{1-\underline{q}\beta\delta}V_0-\frac{1}{\beta\delta(1-\underline{q})}V_1+\frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\frac{1}{\beta\delta(1-\underline{q})}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n}V_{n+1},$$

and from the first order condition we have that

$$\psi_0 = \frac{V_1 - V_0}{1-\beta\delta} + \psi_1 \frac{\beta\delta\left(1-\underline{q}\right)}{1-\beta\delta}.$$

From here, we get that

$$\begin{aligned}
(1-\underline{q})\psi_0 + \eta &= \frac{(1-\underline{q})(V_1 - V_0)}{1-\beta\delta} \\
&+ \frac{\beta\delta\left(1-\underline{q}\right)^2}{1-\beta\delta}\left(\frac{1}{1-\underline{q}\beta\delta}V_0 - \frac{1}{\beta\delta(1-\underline{q})}V_1 + \frac{\underline{q}(1-\beta\delta)^2}{(1-\underline{q}\beta\delta)^2}\frac{1}{\beta\delta(1-\underline{q})}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^n V_{n+1}\right) \\
&= -\frac{(1-\underline{q})}{1-\beta\delta\underline{q}}V_0 + \frac{\underline{q}(1-\beta\delta)}{(1-\underline{q}\beta\delta)}\sum_{n=0}^{\infty}\left(\frac{1-\underline{q}}{1-\underline{q}\beta\delta}\right)^{n+1} V_{n+1} \\
&= \sum_{t=0}^{\infty} p_t^0 V_t - V_0
\end{aligned}$$

which verifies that the value of the dual problem equals the value of the primal. We only need to verify that $\psi_1 \geq 0$ so the constructed multipliers are dual feasible.

$$\begin{aligned}
\psi_1 &= \frac{1}{1-\underline{q}\beta\delta}V_0 - \frac{1}{\beta\delta(1-\underline{q})}V_1 + \frac{(1-\beta\delta)}{\beta\delta(1-\underline{q})^2}\sum_{t=1}^{\infty} p_t V_t \\
&= \frac{1}{1-\beta\delta\underline{q}}\left[V_0 - V_1 + \frac{(1-\beta\delta)}{\beta\delta(1-\underline{q})}\sum_{t=1}^{\infty}\left(\frac{p_t^0}{1-p_0^0}V_t - V_1\right)\right],
\end{aligned}$$

which means that $\psi_1 \geq 0$ if and only if

$$\frac{(1-\beta\delta)}{\beta\delta(1-\underline{q})}\sum_{t=1}^{\infty}\left(\frac{p_t^0}{1-p_0^0}V_t - V_1\right) \geq V_1 - V_0$$

□

F.2.3 Case: $p_{\bar{t}} + p_{\bar{t}+1} = 1$

Finally, we need to consider the case in which for all $0 < t^* \leq \bar{t}$

$$0 \geq \sum_{n=0}^{\infty}\left(\frac{(1-\underline{q})\delta}{1-\underline{q}\beta\delta}\right)^{n+1}\left(u(x_{t^*+n+2}) - \beta u(x_{t^*+n+1}) + (1-\beta)\mathcal{M}(U,\bar{a})\right) \tag{F.17}$$

which means that the conditions in Propositions F.7 and F.9 are not satisfied. In this case, we consider a policy such that the incentive compatibility constraint constraint is binding at time zero,

$p_{\bar{t}} + p_{\bar{t}+1} = 1$ and the incentive compatibility constraint is slack at $\bar{t}, \bar{t}+1$, which yields

$$\psi_0 = \frac{V_{\bar{t}+1} - V_{\bar{t}}}{(\beta\delta)^{\bar{t}} - (\beta\delta)^{\bar{t}+1}} \tag{F.18}$$

$$\eta = V_{\bar{t}} - V_0 - \frac{1 - (\beta\delta)^{\bar{t}}}{(\beta\delta)^{\bar{t}} - (\beta\delta)^{\bar{t}+1}} (V_{\bar{t}+1} - V_{\bar{t}}) \tag{F.19}$$

This means that the probability of monitoring at time $\bar{t}$ is

$$p_{\bar{t}} = \frac{\underline{q} - (\beta\delta)^{\bar{t}+1}}{(\beta\delta)^{\bar{t}} (1 - \beta\delta)}$$

Because the inequality (F.17) is satisfied, we can take $\psi_t = 0$ for all $0 < t < \bar{t}$ and satisfy all the complementary constraints at $0 < t < \bar{t}$. We can pin down the multipliers ψ_0 and η using the first order conditions at time $\bar{t}$ and $\bar{t}+1$. Replacing in the first order conditions at time $\bar{t}+2$, we get that the complementary slackness condition is satisfied for $\psi_{\bar{t}+2} = 0$ if and only if

$$(1 + \beta\delta) V_{\bar{t}+1} - V_{\bar{t}+2} - \beta\delta V_{\bar{t}} = -\left(u(x_{\bar{t}+2}) - \beta u(x_{\bar{t}+1}) + (1 - \beta)\mathcal{M}(U, \bar{a})\right) > 0,$$

which is necessarily the case if

$$0 \geq \sum_{n=0}^{\infty} \left(\frac{(1 - \underline{q})\delta}{1 - \underline{q}\beta\delta} \right)^{n+1} \left(u(x_{\bar{t}+n+2}) - \beta u(x_{\bar{t}+n+1}) + (1 - \beta)\mathcal{M}(U, \bar{a}) \right).$$

Finally, we verify that

$$\begin{aligned}
(1 - \underline{q})\psi_0 + \eta &= (1 - \underline{q}) \frac{V_{\bar{t}+1} - V_{\bar{t}}}{(\beta\delta)^{\bar{t}} - (\beta\delta)^{\bar{t}+1}} + V_{\bar{t}} - V_0 - \frac{1 - (\beta\delta)^{\bar{t}}}{(\beta\delta)^{\bar{t}} - (\beta\delta)^{\bar{t}+1}} (V_{\bar{t}+1} - V_{\bar{t}}) \\
&= \frac{(\beta\delta)^{\bar{t}} - \underline{q}}{(\beta\delta)^{\bar{t}} - (\beta\delta)^{\bar{t}+1}} (V_{\bar{t}+1} - V_{\bar{t}}) + V_{\bar{t}} - V_0 \\
&= p_{\bar{t}} V_{\bar{t}} + p_{\bar{t}+1} V_{\bar{t}+1} - V_0.
\end{aligned}$$

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